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Research Paper

Note on Integral Transform of Fractional Calculus

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ABSTRACT

In recent years Fractional Calculus is highly growing field in research because of its wide applicability and interdisciplinary approach. In this article we study various integral transform particularly Laplace Transform, Mellin Transform, of Fractional calculus i.e. Fractional derivative and Fractional Integral particularly of Riemann-Liouville Fractional derivative, Riemann-Liouville Fractional integral, Caputo's Fractional derivative and their properties.

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1. INTRODUCTION

Fractional Calculus:

It is observed that study of classical calculus i.e. differential calculus and integral calculus begins very close or equally at same period as study of Fractional Calculus begins. Up to 30th September 1695 no one knows that this is the new field of mathematics and called as Fractional Calculus. Answer given by great mathematician Leibnitz on 30th September 1695 to the question raised by another great mathematician L'Hospital as "What will be the meaning of $\frac{d^n f}{dx^n}$, if n=1/2?" as "This is an apparent paradox from which one day a useful consequence will be drawn". After 30th September 1695 mathematicians who are working on classical calculus with special case of differentiation and integration of arbitrary order named the field as Fractional calculus. Though the study of Fractional calculus begins at the end of 17th century but good work held's only since last three to four decades. Oldham and Spanier (1974) [2] and Miller and Ross (1993) [3] gives in brief historical development of Fractional Calculus. Special functions are very closely related with Fractional Calculus and plays important role in development of fractional calculus. [7] Different approaches of Fractional calculus are given throughout the history such Riemann approach, Liouville approach, Caupto approach, Wely approach, Lacorix approach, Grunwald approach, Letnikov approach, Riemann-Liouville approach, Grunwald-Letnikov approach and many more.

Integral Transform:

An integral transform is a particular type of mathematical operator. To each integral transform there is its associated inverse integral transform. This inverse integral transform associated to respective integral transform since, integral transform maps its original domain into another domain where it is solved very easily than in its original domain than again mapped to original domain by using inverse integral transform. There are numerous integral transform in study. [8-10]

Integral Transform of the function f (t) for $t_1 \leq x \leq t_2$ is defined as

$$T(f(t)) = \int_{t1}^{t2} f(t)K(t,x)dt$$

Where K is the function of two variables t and x called kernel of integral transform.

Choice of function K of two variables means kernel is different for different integral transform.

Inverse integral transform is of the form

$$f(t) = \int_{x1}^{x2} T(f(x))\, K^{-1}(x,t)dx$$

Where $K^{-1}(x,t)$ is the kernel of inverse integral transform and is the inverse of kernel K (t, x).

This paper is divided in five sections which are as follows

In section 1, we introduced about fractional calculus and integral transform, in section 2 we see some definitions of fractional calculus, in section 3 some definitions of integral transform are given, in section 4 main result means some integral transform of some fractional approaches are studied with their properties and section 5 is the conclusion part.

2. Some Definitions of Fractional calculus:

Riemann-Liouville Fractional Derivative: [1-6]

Let α > 0 and n-1 < α < n, n ϵ N, and a < x < b, Left hand and Right hand Riemann-Liouville Fractional Derivative is defined as

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt,$$

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_x^b (x-t)^{n-\alpha-1} f(t)\, dt,$$

respectively.

Where, $\Gamma(x) = \int_0^{\infty} t^{x-1}.e^{-t}\ dt$, called as Euler's Gamma function.

If α ϵ N then these definitions acts as like classical derivative of order α.

In general Riemann-Liouville Fractional Derivative of order α of the function f(x) with a < x < b is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt,$$

If 0 < α < 1, we obtain

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt.$$

Riemann-Liouville Fractional Integral:

Let α > 0 and n-1 < α < n , n ϵ N, and a < x < b , Left hand and Right hand Riemann-Liouville Fractional integral is defined as

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt,$$

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t)\, dt,$$

In general Riemann-Liouville Fractional integral of order α ϵ (-∞, ∞) of the function f(x) with a < x < b is defined as

$$I^{\alpha} f(x) = D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt.$$

Caputo Fractional Derivative:

Let α > 0 and n-1 < α < n, n ϵ N, and a < x < b, Left hand and Right hand Caputo Fractional derivative is defined as

$$^{C}D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt,$$

$$^{C}D_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt,$$

In general, Caputo Fractional derivative of order α ϵ (-∞, ∞) of the function f(x) with a < x < b is defined as

$$_{a}^{c}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt,$$

Where n-1 < α < n.

3. Some Integral Transform:

Laplace Transform:

Laplace Transform [8] is very useful in solving differential equations especially when initial values are zero. The Laplace Transform of a function f(x) defined for all real number x ≥ 0 is the function F(s) given by

$\mathcal{L}$(f (t))= $F(s) = \int_0^\infty f(t)e^{-st}$dt,

Inverse Laplace transform of F(s) i.e. f (t) is given by

$\mathcal{L}^{-1}(F(s)) = f(t) = \int_{c-i\infty}^{c+i\infty} e^s$ F(s) ds, c=Re(s) > c_0,

where c_0 lies in right half plane of the absolute convergence of Laplace integral.

Some properties of Laplace Transform:

i) $\mathcal{L}${a f(t)}=a $\mathcal{L}${f(t)} ; where a is constant.

ii) $\mathcal{L}${a f(t) + b g(t)}= a $\mathcal{L}${f(t)} + b $\mathcal{L}$ {g(t)} ; Linearity

iii) If $\mathcal{L}${f(t)}= $F(s)$ then £{f(at)} = $\frac{1}{a} + F(\frac{s}{a})$; change of scale

iv) Convolution property: If F(s) and G(s) are the Laplace Transform of f(x) and g(x) respectively, then

$\mathcal{L}${ F(x) * g(x); s} = F(s) G(s) $=\mathcal{L}\left\{\int_0^x f(x-z)g(z)dz\right\}$,

Where convolution is given by

f * g $=\left\{\int_0^x f(x-z)g(z)dz\right\}$.

v) The Laplace transform of the n-th (n ϵ N) derivative of f(x) is given by

$$\mathcal{L}\{f^{(n)}(x):s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$$

$$= s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0)$$

Mellin Transform:

Mellin Transform [10] is closely related with Laplace transform and Fourier transform. Hjalmar Mellin (1854–1933) gave his name to the Mellin transform 0f a function *f*(*x*) defined over the positive real's, the complex function M [f (x); s]. Mellin Transform is the multiplicative version of two sided Laplace Transform.

The mellin Transform of a function f is defined as

$\mathcal{M}${f(x)} =F(s) $=\int_0^\infty x^{s-1} f(x)\, dx$

Where $\mathcal{M}$ is the the Mellin transform operator and s is the Mellin transform variable
which is complex number.

Inverse mellin transform is defined as

$\mathcal{M}^{-1}\{F(s)\} = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s)\, ds$

Where $\mathcal{M}^{-1}$ is the inverse Mellin transform operator.

Some properties of Mellin Transform:

i) $\mathcal{M}${a f(t)}=a $\mathcal{M}${f(t)} ; where a is constant

ii) $\mathcal{M}${a f(t) + b g(t)}= a $\mathcal{M}${f(t)} + b $\mathcal{M}$ {g(t)} ; Linearity

iii) If $\mathcal{M}${f(t)}= $F(s)$ then $\mathcal{M}${f(at)} $= a^{-s}$ $F(s)$; change of scale

iv) Convolution property : If F(s) and G(s) are the Mellin Transform of f(x) and g(x) respectively, then

a. $\mathcal{M}$ { f(x) * g(x); s} = F(s) G(s)= $\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s-z)g(z)dz\right\}$

b. Where convolution is given by

c. f * g $=\left\{\int_0^x f(x-z)g(z)dz\right\}$.

v) The Mellin transform of the n-th (n ϵ N) derivative of f(x) is given by

$$\mathcal{M}\left\{f^{(n)}(x):s\right\} = (-1)^k \frac{\Gamma(s)}{\Gamma(s-k)} F(s-k)$$

4. Integral transform of some fractional approaches:

5.

Laplace Transform of the Riemann-Liouville Fractional Integral:

The Riemann-Liouville Fractional Integral is given by

$$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt\,.$$

Its Laplace transform is given by

$\mathcal{L}\{D^{-\alpha}f(x)\} = \frac{1}{\Gamma(\alpha)}\mathcal{L}\{x^{\alpha-1}\}\mathcal{L}\{f(x)\} = s^{-\alpha}F(s)$, $\alpha > 0$.

Following table gives Laplace transform of some function and their fractional integrals:

f(x)	F(s)= £{f(x)}	$D^{-\alpha}$\{f(x)\}	£\{$D^{-\alpha}$f(x)\}
e^{ax}	$\frac{1}{s-a}$	$D^{-\alpha}\{e^{ax}\}$	$\frac{1}{s^{\alpha}(s-a)}$
x^{v}	$\frac{\Gamma(v+1)}{s^{v+1}}$	$D^{-\alpha}\{x^{v}\}$	$\frac{\Gamma(v+1)}{s^{\alpha+v+1}}$
$x^{v-1}e^{ax}$	$\frac{\Gamma(v)}{(s-a)^{v}}$	$D^{-\alpha}\{x^{v-1}e^{ax}\}$	$\frac{\Gamma(v)}{s^{\alpha}(s-a)^{v}}$
Cos(ax)	$\frac{s}{s^2+a^2}$	$D^{-\alpha}\{\cos(ax)\}$	$\frac{s}{s^{\alpha}(s^2+a^2)}$
Sin(ax)	$\frac{a}{s^2+a^2}$	$D^{-\alpha}\{\sin(ax)\}$	$\frac{a}{s^{\alpha}(s^2+a^2)}$

Laplace Transform of the Riemann-Liouville Fractional Derivative

The Riemann-Liouville Fractional differential operator is given by

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}(x-t)^{n-\alpha-1}f(t)\,dt\,,$$

The Laplace transform of Riemann-Liouville Fractional differential operator is given by

$$\mathcal{L}\{D^{\alpha}f(x):s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^{k}\left[D^{(\alpha-k-1)}f(0)\right]$$
$$= s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^{n-k-1}\left[D^{k}I^{n-\alpha}f(0)\right]$$

Laplace Transform of the Caputo Fractional Derivative

The Caputo Fractional Differential operator is given by

$${}_{a}^{c}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}f^{(n)}(t)\,dt\,,$$

The Laplace transform of Caputo Fractional Differential operator is given by

$$\mathcal{L}\{{}_{a}^{c}D_{x}^{\alpha}f(x):s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0),\ \ n-1 < \alpha < n\,.$$

Mellin Transform of the Riemann-Liouville Fractional Integral:

Mellin transform of Riemann-Liouville fractional integral operator is given by

$$\mathcal{M}\{D^{-\alpha}f(x)\} = F(s) = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)}F(s+\alpha)$$

Mellin Transform of the Riemann-Liouville and Caputo Fractional Derivative:

Mellin transform of Riemann-Liouville and Caputo fractional derivative operator is given by

$$\mathcal{M}\{D^{\alpha}f(x)\} = F(s) = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}F(s-\alpha)$$

6. CONCLUSION

Applications of Fractional calculus are found in almost all sciences. Fractional Calculus is the topic of today's researchers. Fractional calculus is termed as a solution to real world problems. Integral transform are useful in finding solution of differential equations. Solving fractional differential equation is very tedious task. Extending integral transform to solve fractional differential equations is the target to achieve.

In this paper we study different approaches of fractional calculus and their properties.

Also we study integral transform with their properties and the integral transform particularly Laplace transform and Mellin transform of Riemann-Liouville integral operator, Riemann-Liouville and Caputo differential operator.

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