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Research Paper

A Note on Existence and Uniqueness of Nearest Points in a Set

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ABSTRACT

In this paper, the main problem is concerned with the existence and uniqueness of nearest points of a given point in a set with *P*- property and weak *P*- property.
Key words: Proximinal set, *P*- property, weak *P*- property, Metric projection

1. INTRODUCTION

The traditional best approximation theory concentrates on the distance between a point and a convex set. The interest in best approximation problem is common to several branches of mathematics, such as Approximation theory, Functional analysis, Convex analysis, Optimization, Numerical Linear Algebra, Statistics, and other fields. The distance function is used to define the distance between a point and a set, the distance between two sets and the diameter of a set. The basic terms related to this problem are proximal set which was proposed by Killgrove and used first by Phelps. More than 40 years ago, Ivan Singer [8] in Problem 2.1, asked whether every infinite-dimensional Banach space contains proximinal subspaces of co-dimension 2. Recently Charles J. Read [7] answered this question in the negative. The term Chebyshev set was introduced by Stechkin in honor of the founder of best approximation theory. In the middle of previous century Chebyshev proved that every subspace of C[0, 1] of polynomial of degree n and subset $\mathbb{R}_{m,n}$ of rational fractions

$(a_0+a_1x_1+ \cdots +a_n x_n)/(a_0+a_1x_1 \cdots +a_n x_m)$

with fixed $m, n \in\mathbb{N}$ are Chebyshev subsets.

In [2] a class of reflexive non-Kadec Klee norms is exhibited for which some nearest points always exist. The Lau-Konjagin Theorem (see [2]) states that in a reflexive space, for every closed set M there is a dense (or generic) set in X which admits nearest points if and only if the norm has the Kadec-Klee property.

Definition 1.1. Let M be a nonempty set in a metric space, (X, d) and for any $x\in X \setminus M$ define

$$d(x, M) = \inf \{d(x, y) : y \in M\} \qquad (1)$$

The function $d(\cdot, M) : X \to [0,\infty)$ is called the distance function associated to M. If there is m in M such that $d(x, m) = d(x, M)$, i. e., m in M has minimum distance from given x, then m is called nearest point to x in M. We face with the following problem:

Problem 1.1(a) Under what conditions do we have m in M such that $d(x, m) =d(x, M)$?
(b) If such an m in M exists, is it unique?

In this paper we will discuss these questions. The following well-known result asserts that if M is a complete convex subset of an inner product space X, then each $x \in X$ has a unique element of best approximation in M.

Theorem 3.3 (Minimizing vector) [12] Let M be a nonempty complete convex set in an inner product space X. Then M is a Chebyshev set, that is,

$$\forall x \in X \; \exists! \; y \in M : ||x - y|| = \inf_{\bar{y} \in M} ||x - \bar{y}||.$$

In this paper, we will generalize this result.

2. Proximinal sets

The set valued mapping $P_M : X \to \mathcal{P}(X)$ is defined by

$$P_M(x) = \{m \in M: d(x, m) = d(x, M)\}.$$

The set P_M is called the metric projection of x in M. If $P_M(x) \neq \emptyset$ for all $x \in X$, then M is called a proximinal set or a set of existence. If $P_M(x)$ contains at most one element for all $x \in X$, then M is called a set of uniqueness. If $P_M(x)$ contains exactly one element for all $x \in X$, then M is called a Chebyshev set.
In view of the above definitions, we can reformulate Problem 1.1 as follows:

Problem 2.1 (a) Under what conditions is M proximinal, i.e., $P_M(x) \neq \emptyset$ for all $x \in X$?
(b) When is M Chebyshev, i.e., $P_M(x)$ is a singleton set?

The following examples show that the nearest point may or may not exist and if it exists, it may not be unique.
Example 2.2. Consider $M = \mathbb{R}^2 \setminus B(0,1) \subseteq \mathbb{R}^2$ equipped with the Euclidean norm. It is easy to check that for any $x \in B(0,1) \setminus \{0\}$,

$$P_M(x) = \{x/||x||\}.$$

However, $P_M(0) = \{y \in \mathbb{R}^2: ||y|| = 1\}$. Hence M is proximinal, but not a Chebyshev set.

Example 2.3. Let $M = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$. If $x \in \mathbb{R}^2 \setminus M$, then the foot of the perpendicular drawn from x to the x-axis is the nearest point in M to x and it is unique. Therefore, M is a Chebyshev set.

Example 2.4. Consider $M = \mathbb{R}^2 \setminus B[0,1]$, $\mathbb{R}^2$ being equipped with the Euclidean norm. Any point $x \in B[0,1] = \mathbb{R}^2 \setminus M$ has no nearest point in M, because M is open. Hence M is not proximinal.

3. Continuity of the Distance function [10]

Theorem 3.1 Let X be a metric space, $M \subseteq X$ and $x \in X$. Then the distance function $D_x : u \mapsto d(x, u)$ is nonexpansive (and so continuous on M).

Proof. Let $x \in X$ be fixed. If $u, v \in M$, then

$$d(x, u) \leq d(x, v) + d(u, v)$$
$$\Rightarrow d(x, u) - d(x, v) \leq d(u, v).$$

By symmetry, we have

$$d(x, v) - d(x, u) \leq d(u, v).$$
$$\therefore \quad | d(x, u) - d(x, v)| \leq d(u, v).$$

That is, $| D_x(u) - D_x(v)| \leq d(u, v)$.

Hence the distance function D_x is non-expansive (and so continuous on M).

4. Existence in the case of a Compact Set

Theorem 4.1(*Extreme Value Theorem*). Let X be compact and $f : X \to \mathbb{R}$ be continuous function. Then f takes on a maximum value and a minimum value on X, that is, there exist $a, b \in X$ such that

$$\forall \; x \in X \; f(a) < f(x) < f(b).$$

Theorem 4.2 Let M be a subset in a metric space X and $x \in X$. If M is compact, then

$$\exists \; v \in M : d(x, v) = d(x, M).$$

Proof. The distance function $D_x: M \to \mathbb{R}$, given by

$$D_x : u \mapsto d(x, u) \; (u \in M)$$

is continuous(see Theorem 3.1) and by assumption the set M is compact. Therefore the Extreme Value Theorem applies and indicates that this greatest lower bound is realized at some point of M (D_x takes on a minimum value on M) That is, there exist $v \in M$ such that

$$d(x, v) = \inf \{d(x, u) : u \in M \} = d(x, M).$$

Without invoking the Extreme Value Theorem, we can prove the same result. For this we need the notion of minimizing sequence.

Definition 4.3. A sequence $\{zn\}$ of elements in M is called a minimizing sequence in M for x if

$$d(x, M) = \lim_n d(x, z_n).$$

Theorem 4.4. Suppose that M is a closed set in a metric space X and $x \in X \backslash M$. If some minimizing sequence $\{z_n\} \subseteq M$ for x has a limit point $z \in M$, then z is a nearest point to x in M.

Proof. Let $\{z_n\} \subseteq M$ be a minimizing sequence for x, that is,

$$d(x, z_n) \rightarrow d(x, M).$$

If it has a limit point $z \in M$, then passing to a subsequence we may assume that

$$d(z, z_n) \rightarrow 0.$$

We then have

$$\begin{aligned} d(x, M) \leq \mathrm{d}(x, z) &\leq d(x, z_n) + d(z_n, z) \\ &\rightarrow d(x, M) + 0 \\ &= d(x, M) \\ \Rightarrow d(x, z) &= d(x, M). \end{aligned}$$

Therefore, z is a nearest point to x in M.

Corollary 4.5. Let M be a compact set in a metric space X. Then, every $x \in X$ has a nearest point in M.

Proof. Let $\{z_n\} \subseteq M$ be a minimizing sequence for x, that is,

$$d(x, z_n) \rightarrow d(x, M).$$

By the compactness of M, this sequence has at least one limit point z in M. Hence by the previous theorem, $z \in M$ is a point nearest to x.

5.Uniqueness in the case of a Compact Set

For the uniqueness of the nearest point, we have to impose an additional condition on X.

Definition 5.1 [1] Let (A, B) be a pair of nonempty subsets of a metric space(X, d). The pair (A, B) is said to have P-property if

$$d(x, u) = d(y, v) = d(A, B) \Rightarrow d(x, y) = d(u, v)$$

where $x, y \in A$ and $u, v \in B$.

Definition 5.2. A metric space X is said to have P-property if every pair of nonempty closed sets in X has P-property.

Theorem 5.3. Let M be a set in a metric space X and $x \in X$. If M is compact and X has P-property, then

$$\exists!\, v \in M : d(x, v) = d(x, M).$$

Proof. In Theorem 4.2, we have shown that

$$\exists\, v \in M : d(x, v) = d(x, M).$$

For the uniqueness, we observe that x and M are closed sets in X. Since X has P-property, we have

$$d(x,u)= d(x,v)= \mathrm{dist}(A,B) \Rightarrow d(u,v)=d(x,x)=0.$$

taking $A = \{x\}$, $B = M$ with $u, v \in B$ in the definition of P-property. Thus,

$$d(u, v) = 0 \Rightarrow u = v.$$

The uniqueness result can be obtained under more weaker condition on X. In [5] the P-property has been weakened to the weak P-property.

Definition 5.4 [3,6] Let (A, B) be a pair of nonempty closed subsets of a metric space

(X, d) with $A_0 \neq \phi$. The pair (A, B) is said to have weak P-property if

$$d(x,u)=d(y,v) = \text{dist}(A, B) \Rightarrow d(x, y) \leq d(u,v)$$

where $x ,y \in A$ and $u, v \in B$.

Definition 5.6. A metric space X is said to have weak P-property if every pair of nonempty closed sets in X has weak P-property.

The next result generalizes Theorem 5.3.

Theorem 5.7. Let M be a set in a metric space X and x in X. If M is compact and X has weak P-property, then

$$\exists!\, v \in M : d(x, v) = d(x, M).$$

Proof. In Theorem 4.2, we have shown that

$$\exists\, v \in M : d(x, v) = d(x, M).$$

For the uniqueness, we observe that M and $\{x\}$ are closed sets in X. Since X has weak P-property, we have

$$d(u, x) =d(v,x) =\text{dist}(A,B)$$
$$\Rightarrow d(u,v) \leq d(x,x)=0.$$

taking $A = M$, $B = \{x\}$ with $u, v \in A$ in the definition of P-property. Thus,
$d(u, v) = 0 \Rightarrow u = v$.

6.Uniqueness in the case of Weak Compactness

Theorem 6.1. Let X be a Banach space. If M is non-empty and boundedly weakly compact and X has the weak P-property, then M is Chebyshev.

Proof. Suppose that $x \in X \setminus M$ and let $\{zn\}$ be a minimizing sequence in M for x. Then $\{zn\}$ lies $M \setminus B[0, r]$ for some $r > 0$, and so has a weak cluster point z belonging to M. By Theorem 4.4, z is a nearest point to x.

7. Closed and Convex set

Theorem 7.1 [11] A Banach space X is reflexive if and only if each closed convex subset (or each closed subspace) M of X is proximinal.

Proof. The necessity condition is easily deduced, for instance, from the weak compactness of the unit sphere in such a space. Less well known, however, is the truth of the converse. If X is a non-reflexive Banach space it must contain a separable nonreflexive subspace M. James [4] has shown that if a separable Banach space is nonreexive, then there exists a linear functional on the space which does not attain its supremum on the unit sphere. Hence, there exists $f\in S(M)^*$ such that $f^{-1}(1)$ misses $S \cap M$. Considered as a subset of X, $f^{-1}(1)$ is a closed convex set which is not proximinal (the origin has no nearest point in $f^{-1}(1)$, which completes the proof.

Theorem 7.3. [8,9] Let $(X, \|\cdot\|)$ be a reflexive Banach space and X is strictly convex. Then every nonempty closed convex subset $M \subseteq X$

$$\exists!\, u \in M : \|x - u\| = d(x, M).$$

Since uniformly convex and uniformly smooth Banach spaces are reflexive and strictly convex, the above theorem implies that if $(X, \|\cdot\|)$ is a uniformly convex and uniformly smooth Banach space, then every nonempty closed convex subset $M\subseteq X$ is a Chebyshev set. Furthermore, as a special case of uniformly convex and uniformly smooth Banach space, every nonempty closed convex subset of a Hilbert space is a Chebyshev set.

Theorem 7.4. Let X be a reflexive Banach space with P-property and M be a nonempty, closed, and convex subset of X. Then for any fixed $x \in X$ there exists a unique $u\in M$ such that

$$\|x - u\| = \inf_{m \in M} \|x - m\|.$$

Proof. In Theorem 4.2, we have shown that

$\exists u \in M : ||x - u|| = d(x, M)$.

For the uniqueness, we observe that $\{x\}$ and M are closed sets in X. Since X has P-property, we he

$$||x - u||=||x - v||=\mathrm{d}(A, B)\Rightarrow||u - v||=||x - x|| =0.$$

taking $A = \{x\}$, $B = M$ with u, $v \in B$ in the definition of P-property. Thus,

$$||u - v|| = 0 \Rightarrow u = v.$$

The uniqueness of Theorem 4.2 is also true under weaker condition on X.

Theorem 7.5. Let M be a nonempty closed convex subset of a reflexive Banach space X and $x \in X$. If X has weak P-property, then

$$\exists!\ u \in M : ||x - u|| = d(x, M).$$

Proof. In Theorem 4.2, we have shown that

$$\exists!\ u \in M : ||x - u|| = d(x, M).$$

For the uniqueness, we observe that M and $\{x\}$ are closed sets in X. Since X has weak P-property, we have

$$||x - u||=||x - v||=d(A, B)\Rightarrow||u - v||\le||x - x||=0,$$

taking $A = \{x\}$, $B = M$ with u, $v \in B$ in the definition of P-property. Thus,

$$||u - v|| = 0 \Rightarrow u = v.$$

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