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Research Paper

The Strongly Compact Algebras

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ABSTRACT

Algebra of bounded linear operators on a Hilbert space is strongly compact if its unit ball is relatively compact in the strong operator topology. The algebra generated by the operator and the identity is strongly compact. This notion was introduced by Lomonosov as an approach to the invariant subspace problem for essentially normal operators. The basic properties of strongly compact algebras are established. A characterization of strongly compact normal operators is provided in terms of their spectral representation, and some applications. Finally, necessary and sufficient conditions for a weighted shift to be strongly compact are obtained in terms of the sliding products of its weights, and further applications are derived.

Key words: strongly compact, Hilbert space, operators, subalgebra, relatively, commutant, topology.

1. INTRODUCTION

Let $B(H)$ denote the algebra of all bounded linear operators on a Hilbert space H. We say that a subalgebra$\mathcal{R}$of $B(H)$ is strongly compactif the unit ball of $\mathcal{R}$ is relatively compact in the strong operator topology. Also, we say that an operator Tin $B(H)$is strongly compactif the algebra generated by $T$and $I$is strongly Compact. This notion was introduced by Lomonosov [1] as an approach to the invariant subspace problem for essentially normal operators. Recall that an operator Tin $B(H)$ is essentially normal if and only if the operator $T^*T - TT^*$is compact. Also, recall that the commutant of an operator is the algebra of alloperators that commute with the given operator. Lomonosov [1] proved that if $T \in B(H)$ is an essentially normal operator such that neither the commutant of Tnor the commutant of $T^*$are strongly compact algebras then Thas a non-trivial invariant subspace.Therefore, it is an interesting task to study the structure of strongly compact algebras, because a better understanding of such algebras would give more insight on the invariant subspace problem for essentially normal operators.

A classification of self-adjoint, strongly compact algebras was carried out by Marsalli, [2] who showed that a self-adjoint sub algebra of $B(H)$ is strongly compact if and only if it can be written as a direct sum of finite-dimensional, self-adjoint sub algebras of $B(H)$. Thus, every self-adjoint strongly compact operator is unitarily equivalent to a diagonal operator. We will see later that, in general, a strongly compact normal operator is not unitarily equivalent to a diagonal operator.

Let T be an operator on a Hilbert space H and let $E \subseteq H$ be an invariant subspace of$T$. There are two operators associated to $T$and E, namely, the restriction operator $T_{/E} \in B(E)$defined by $T_{/E}\, x = Tx$ for every$x \in E$, and the quotient operator$\tilde{T} \in B(H/E)$ defined by $\tilde{T}(x + E) = Tx + E$for every$x + E \in H/E$. It is natural to ask whether the restriction and the quotient of a strongly compact operator are strongly compact, too. We construct counterexamples to both questions.

In this study organized as follows. First of all, in this section we present two different examples of strongly compact algebras and we give a procedure for constructing new strongly compact algebras out of old ones. Next, in this section we provide a characterization of strongly compact normal operators in terms of their spectral representation. Finally, in this section we obtain necessary and sufficient conditions for a weighted shift with non-zero weights to be strongly compact in terms of the sliding products of its weights.

As a warm-up, it will be convenient to state two characterizations of compactness, one for the norm topology of $H$and the other one for the strong operator topology of $B(H)$

Lemma 1. A subset S of H is relatively compact if and only if for every $\varepsilon > 0$ there is a compact subsetK_ε *of* Hsuch that$S \subseteq K_\varepsilon + \varepsilon B_H$.

This characterization can be found in Diestel's book. [3] The non-trivial part of its proof relies on the fact that any totally bounded subset of H is relatively compact.

Lemma 2. *Let S be a bounded subset of$B(H)$. Then the following conditions are equivalent.*

(i) *S is relatively compact in the strong operator topology.*

(ii) *$\{Tx : T \in S\}$ is relatively compact for all $x \in H$.*

(iii) *There is a dense subset D of H such that $\{Tx : T \in S\}$ is relatively compact for all $x \in$D*The equivalence of (i) and (ii) was noticed by Marsalli [2] and it does not need the assumption that Sbe bounded because of the Uniform Boundedness Principle. The implication (i)⇒(ii) follows from the continuity of the map $T \longrightarrow Tx$with respect to the strong operator topology. The implication (ii) ⇒(iii) is trivial. The implication (iii) ⇒ (i) is a consequence of Tychonov's Theorem on the product of compact spaces in combination with Lemma 1.

Recall that a vector $x_0 \in H$is said to be *cyclic* for a subalgebra $\mathcal{R}$ of $B(H)$ if the orbit $\{Rx_0 : R \in \mathcal{R}\}$is dense in H. A vector $x_0 \in H$is said to be *cyclic* for an operator$T \in B(H)$ whenever x_0 is cyclic for the subalgebra generated by $T$and I in $B(H)$.

A straightforward consequence of Lemma 2 is that a subalgebra R of $B(H)$ with a cyclic vector$x_0 \in H$ is strongly compact if and only if the set$\{Rx_0 : R \in \mathcal{R}, \|R\| \leq 1\}$is relatively compact. This fact will be used later in the proofs of Theorem 4 and Theorem 5.

2. Basic properties of strongly compact algebras

In this section we present two different examples of strongly compact algebras. These examples are the bricks for building strongly compact algebras. We also give a procedure for constructing new strongly compact algebras out of old ones.

Proposition 1. *Let H be an infinite-dimensional Hilbert space, and let$(E_i)_{i\in I}$be a family of* finite-dimensional *subspaces of H whose union is dense in H. If $\mathcal{R}$ is a subalgebra of $B(H)$with the property that E_i is invariant under R for all $R \in \mathcal{R}$ and $\in I$, then $\mathcal{R}$is strongly compact.*

Proof. Consider the dense subset $D$of Hgiven by

$$D = \bigcup_{i \in I} E_i$$

If $x \in D$then $x \in E_i$for some $i \in I$.Since$E_i$is invariant under $R$for every $R \in \mathcal{R}$we have

$\{Rx_0 : R \in \mathcal{R}, \|R\| \leq 1\} \subseteq \|x\|. B_{E_i}$

It follows that the algebra Rsatisfies the condition (iii) in Lemma 2 and therefore it is strongly compact.

An application of Proposition 1 is the following. Let Hbe a separable, infinite-dimensional Hilbert space, let $(e_n)_{n\geq 0}$ be an orthonormal basis of H, and let Bdenote the backward shift on H, that is $Be_0 = 0$ and $Be_n = e_{n-1}$ for $n \geq 1$. Put E_n= span $\{e_0, e_1, \ldots, e_n\}$. It is clear that the union of the E_n's is dense in H and that every E_n is invariant under B. It follows that the backward shift is a strongly compact operator.

Another situation to which Proposition 1 applies is the case of a compact normal operator, since it is a well-known fact that every compact normal operator has an orthogonal basis of eigenvectors. In general, any operator on $H$with lower triangular matrix with respect to an orthogonal basis $(e_n)_{n\geq 0}$is strongly compact.

Proposition 2. The commutant $\mathcal{R}$ of a compact operator K with dense range is strongly compact.

Proof. Let $D = KH$ and let $y \in D$,say $y = Kx$. Then

$$\{Ry : R \in \mathcal{R}, \|R\| \leq 1\} = \{RKx : R \in \mathcal{R}, \|R\| \leq 1\} = \{KRx : R \in \mathcal{R}, \|R\| \leq 1\} \subseteq \|x\|. KB_H$$

and again it follows from Lemma 2 that $\mathcal{R}$ is strongly compact.

Although the above examples may overlap, they are essentially different, since the backward shift does not commute with a non-zero compact operator. The proof goes by contradiction. Suppose $BK = KB$for some non-zero compact operator$K$. Then$K^*$is also a non-zero compact operator and we have$K^* (B^*)^n = (B^*)^n K^*$for all $n \geq 0$. Fix an$n_0 \geq 0$ such that $K^* e_{n0} \neq 0$. Then

$(B^*)^n e_{no} = e_{n+n0} \to 0$weakly as $n \to \infty$,

So that$\|K^*(B^*)^n e_{no}\| \to 0$ as $n \to \infty$.But $(B^*)^n$ is an isometry for all $n_0 \geq 0$, so that

$$\|(B^*)^n K^* e_{no}\| = \|K^* e_{no}\| > 0$$

for all $n \geq 0$, which is a contradiction.

It is worth-while to observe that the assumption that Khas dense range cannot be removed from Proposition 2. This will be shown in Proposition 8 below.

The next result gives a procedure for constructing new strongly compact algebras out of old ones.

Proposition 3. *The direct sum of a family of strongly compact algebras is also strongly compact*

Proof. Let $\{H_i : i \in I\}$ be a family of Hilbert spaces and consider the direct sum

$$H \bigoplus_{i \in I} H_i = \{x = (x_i) : x_i \in H_i, \|x\|^2 = \sum_{i \in I} \|x_i\|^2 < \infty\}.$$

Let $\mathcal{R}_i$be a strongly compact subalgebra of $B(H_i)$ for every$i \in I$and look at the direct sum

$$\mathcal{R} = \bigoplus_{i \in I} R_i = \{T = (T_i) : T_i \in \mathcal{R}_i, \|T\| = sup_{i \in I} \|T_i\| < \infty\}.$$

We have to show that $\mathcal{R}$ is a strongly compact subalgebra of $B(H)$. We will check that its unit ball satisfies condition (ii) of Lemma 2. Fix a vector$x = (x_i) \in H$, let $\varepsilon > 0$, and choose a finite set $I_0 \subseteq I$such that

$$\sum_{i \in I \setminus I_0} \|x_i\|^2 < \varepsilon^2$$

Consider the vector $\mathcal{Z} \in H$defined by $\mathcal{Z}_i = x_i$ if $i \in I_0$ and $\mathcal{Z}_i = 0$ otherwise. Since every $\mathcal{R}_i$is strongly compact and I_0 is finite we have that$K_\varepsilon = \{TZ : T \in \mathcal{R}, \|T\| \leq 1\}$ *is* a relatively compact subset of$H$. Also, $\{Tx : T \in \mathcal{R}, \|T\| \leq 1\} \subseteq K_\varepsilon + \varepsilon . B_H$. It follows from lemma 1 that the set $\{Tx : T \in \mathcal{R}, \|T\| \leq 1\}$ *is* relatively compact in H .

Notice that the algebra generated by the direct sum of a family of operators and the identity is a subalgebra of the direct sum of the algebras generated by the operators and the identity. Hence, as a consequence of Proposition 3, the direct sum of a family of strongly compact operators is also a strongly compact operator.

The converse of Proposition 3, is false. In section 3, we will construct an example of a family of operators $\{T_i : i \in I\}$such that $T_i$fails to be strongly compact for every$i \in I$although the direct sum is a strongly compact operator.

3. Strongly compact normal operators

Recall that an operator Non a Hilbert space is said to be normal if $N^*N = NN^*$. An example of a normal operator can be constructed as follows. Let $\Omega \subseteq \mathbb{C}$ be a compact set and let μbe a finite measure defined on the Borel subsets of Ω .Then the operator $M_\mathcal{Z}$of multiplication by $\mathcal{Z}$defined on $L^2(\mu)$ by $M_\mathcal{Z} f = \mathcal{Z} f(\mathcal{Z})$for every $f \in L^2(\mu)$is normal.

One version of the Spectral Theorem [6, p.15] can be stated as follows. If $N$is a normal operator defined on a separable Hilbert space, then there exist a finite or countable family $(\mu_n)_{n \geq 1}$ of probability measures on a compact subset of $\mathbb{C}$, and a unitary operator $U:\bigoplus_{n\geq 1} L^2(\mu_n) \to H$ such that $U^*NU = \bigoplus_{n\geq 1} T_n$, where T_n is the operator of multiplication by $\mathcal{Z}$on $L^2(\mu_n)$. Hence, N is strongly compact if and only if so is $\bigoplus_{n\geq 1} T_n$.

We start with a characterization of those measures $\mu$for which the operator $M_z$of multiplication by$\mathcal{Z}$is strongly compact on $L^2(\mu)$. Let πdenote the space of all polynomials $p(\mathcal{Z})$ in one complex variable provided with the norm $\|p\|_\infty in L^\infty(\mu)$.

Theorem 1. *The following conditions are equivalent:*

(i) M_Z*is strongly compact.*

(ii) *The natural embedding* $J : \pi \to L^2(\mu)$*is compact.*

(iii) *Any bounded sequence in* π *has* aμ-*almost everywhere convergent subsequence.*

Proof. Let $p$be any polynomial and notice that $p(M_Z) = M_{p(Z)}$ where $M_{p(z)}$denotes the operator ofmultiplication by$p$. It is easy to check and it is shown in Halmos's book [5] that the norm of the operator $M_f$of multiplication by f is the norm of f in $L^\infty(\mu)$ and so $\|pM_z\| = \|p\|_\infty$.

(i) $\Longrightarrow$ (ii) : Assume M_z is strongly compact on $L^2(\mu)$. Then

$\{p \in \pi : \|p\|_\infty \leq 1\} = \{p(M_z).1 : p \in \pi, \|p(M_z)\| \leq 1\}$

is a relatively compact subset of $L^2(\mu)$ and therefore J is compact.

(ii) $\Longrightarrow$(iii): Let$(p_n)_n$ be a bounded sequence in π Since $J$is compact, there is a subsequence $(p_{n\mathrm{j}})$jthat converges in $L^2(\mu)$to g, say. Hence, there is a subsequence $\left(p_{n_{j_k}}\right)_k$that converges μ -almost everywhere to g.

(iii) $\Longrightarrow$(i): Let $f \in L^2(\mu)$and let us show that

$\{p(M_z)f : p \in \pi, \|p(M_z)\| \leq 1\} = \{p.f : p \in \pi, \|p\|_\infty \leq 1\}$

is a relatively compact subset of $L^2(\mu)$. Let $(p_n)_n$be a sequence with $\|p_n\|_\infty \leq 1$, for all $n$and take a subsequence$(p_{nk})_k$that converges μ-almost everywhere to g, say. Then$|p_{nk}.f| \leq |f|\mu$ -almost every where.

It follows from the Bounded Convergence Theorem that$\|p_{nk}.f - g.f\|_2 \to 0$as $k \to \infty$.

As an application of Theorem 1we present two examples to show that the operator$M_Z$of multiplication by $\mathcal{Z}$may or may not be strongly compact.

Corollary 1. *(V. Shulman, private communication.) Let* $\mathbb{D}$*denote the open unit disc provided with thetwo-dimensional Lebesgue measure. Then the operator* M_Z *of multiplication by* $\mathcal{Z}$*on* $L^2(\mathbb{D})$ *is strongly compact.*

Proof. If $(p_n)_n$is a sequence of polynomials with $\|p\|_\infty \leq M$for all $n$then Montel's Theorem gives asubsequence $\left(p_{n_k}\right)_k$that converges uniformly on compact subsets of $\mathbb{D}$and so almost everywhere.

Since $M_Z$has no eigenvectors, Corollary 1provides an example of a strongly compact normal operator that is not unitarily equivalent to a diagonal operator.

Recall that the Bergman space is the subspace $A^2(\mathbb{D})$ of all functions in $L^2(\mathbb{D})$which are analytic on$\mathbb{D}$. The above argument shows that multiplication by $\mathcal{Z}$is also a strongly compact operator on the Bergman space. We will come back to this fact in section 4.

Corollary 2. *Let* $[0,1]$ *be the unit interval provided with the one-dimensional Lebesgue measure. Then the operator* M_x*of multiplication by* x *on* $L^2[0, 1]$ *fails to be strongly compact.*

Proof. Let $C[0, 1]$ denote the space of continuous functions on [0, 1] provided with the sup norm. The Stone-Weierstrass Theorem ensures that $\pi$is dense in C [0,1]. Therefore it suffices to show that the natural embedding $J : C[0, 1] \to L^2[0, 1]$ fails to be compact. Consider the sequence of functions defined by$f_n(T) = \cos 2\pi nt$. Then $(f_n)_n$is bounded in C [0, 1] and it has no convergent subsequence in $L^2[0, 1]$.

Proposition 4. *Let* T *be any operator on* H *such that* $\|T\| <1$ *and let* T_I*be any norm-one strongly compact operator on a Hilbert space* H_I *, whose spectrum is equal to the* *closure of $\mathbb{D}$. Then the operator $T \oplus T_I$ is strongly compact on $H \oplus H_I$.*

Proof. Let p be any polynomial and apply von Neumann's inequality to get $\|p(T)\| \leq \|p\|_\infty = \|p(T_I)\|$, so that

$$\|p(T \oplus T_I)\| = \max\{\|p(T)\|, \|p(T_I)\|\}$$
$$= \|p\|_\infty.$$

A proof of von Neumann's inequality can be found in Halmos's book. [5] Since the set $\{p(T_I): \|p\|_\infty \leq 1\}$ is relatively compact in the strong operator topology of $B(H_I)$ it suffices to prove that the set $\{p(T): \|p\|_\infty \leq 1\}$ is relatively compact in the strong operator topology of $B(H)$. We will show that condition (ii) of Lemma 2is satisfied. Fix a vector $x \in H\backslash\{0\}$, , let $\varepsilon > 0$ and choose an n_0 such that

$$\sum_{n=n_0+1}^{\infty} \|T\|^n < \frac{\varepsilon}{\|x\|}$$

let $E = \text{span}\{x, Tx, \dots, T^{n_o}x\}$, put $M = \|x\| + \|Tx\| + \cdots + \|T^{n0}x\|$, and consider the compact set $K_\varepsilon = M.B_E$. take a polynomial $p(z) = a_0 + a_1 z + \cdots + a_N z^N$ with $\|p\|_\infty \leq 1$ and notice that $|a_n| \leq 1$ for every $0 \leq n \leq N$. Thus, $a_0 x + a_1 Tx + \cdots + a_{n_o} T^{n0} x \in K_\varepsilon$. On other hand $(a_{n_o+1} T^{no+1} + \cdots + a_N T^N)x \in \varepsilon.B_H$.

Hence , $p(T)x \in K_\varepsilon + \varepsilon . B_H$. Now it follows from lemma 1that $\{p(T)x : p \in \pi, \|p\|_\infty \leq 1\}$ is a relatively compact subset of H.

Notice that the operator T in Proposition 4need not be strongly compact. Thus, the direct sum of two operators may be strongly compact when one of the summands is not. Later on we will provide a stronger version of this result, namely, none of the summands will be strongly compact. For the time being we have an example of a strongly compact operator with a restriction and a quotient that fail to be strongly compact.

Recall that two measures are equivalent if they have the same null sets, and they are orthogonal if they are concentrated on disjoint sets. In the sequel we will use the following result.

Lemma 3. *Let μ_1 and μ_2 be two finite measures on a compact set $\Omega \subseteq \mathbb{C}$, and let T_i denote the operatorof multiplication by z on $L^2(\mu_i)$ for i= 1, 2.*

(i) If μ_1 and μ_2 are equivalent then T_1 and T_2 are unitarily equivalent.

(ii) *If μ_1 and μ_2 are orthogonal then the operator of multiplication by z on $L^2(\mu_1 + \mu_2)$ is unitarily equivalent to $T_1 \oplus T_2$.*

Proof. (i) Let h denote the Radon-Nikodym derivative of μ_1 with respect to μ_2 and define

$$U: L^2(\mu_1) \to L^2(\mu_2)$$
$$f \mapsto f\sqrt{h}.$$

It is easy to check that U is a unitary operator and $T_1 = U^* T_2 U$.

(ii) Let B_1 and B_2 be two disjoint Borel subsets of Ω such that for i= 1, 2 the measure μ_i is concentrated on B_i, that is , $\mu_i(\Omega\backslash B_i) = 0$. Define

$U : L^2(\mu_1) \oplus L^2(\mu_2) \to L^2(\mu_1 + \mu_2)$

$(f_1, f_2) \mapsto f_1 \mathrm{X} B_1 + f_2 \mathrm{X} B_2$.

It is easy to check that U is a unitary operator and that $U(T_1 \oplus T_2)U^*$ is the operator of multiplication by $\mathcal{Z}$ on $L^2(\mu_1 + \mu_2)$.

Part (ii) of Lemma 3can be easily generalized to a countable family of measures as follows.

Lemma 4. *Let $((\mu_n)_n)$ be a sequence of pairwise orthogonal probability measures on a compact set $\Omega \subseteq \mathbb{C}$, let T_n denote the operator of multiplication by $\mathcal{Z}$ on $L^2(\mu_n)$, and put $\mu = \sum_{n=1}^{\infty} \mu_n / 2^n$ .Then the operatorof multiplication by $\mathcal{Z}$ on $L^2(\mu)$ is unitarily equivalent to the operator $\oplus_{n=1}^{\infty} T_n$*

Proof. For every n , let h_n denote the Radon-Nikodym derivative of μ_n with respect to μ. Observe that thesets $\{\mathcal{Z} \in \Omega : h_n(\mathcal{Z}) \neq 0\}$ are pairwise $\mu-$ almost disjoint and therefore we can define

$$U: \oplus_{n=1}^{\infty} L^2(\mu_n) \longrightarrow L^2(\mu)$$
$$(f_n)_n \longmapsto \sum_{n=1}^{\infty} f_n \sqrt{h_n} .$$

It is easy to check that U is a unitary operator and that $U(\oplus_{n=1}^{\infty} T_n)U^*$ is the operator of multiplication by z on $L^2(\mu)$.

The next result shows that the summands of a strongly compact direct sum need not be strongly compact. Moreover, all the operators involved in the construction are normal.

Proposition 5. *Let $(t_n)_{n\geq 1}$,be a sequence of distinct points dense in* $[0,2\pi)$, *let μ_n denote the one-dimensional Lebesgue measure on the radial segment $I_n = (0, e^{it_n}]$ and let $T_n$denote the operator of multiplication by$\mathcal{Z}$on $L^2(\mu_n)$. Then T_n fails to be strongly compact for every $n \geq 1$ and the operator $\oplus_{n=1}^{\infty} T_n$is strongly compact.*

Proof. For every, it is easily seen that T_n is unitarily equivalent to the operator$e^{it_n} M_x$where$M_x$isthe operator of multiplication by $x$on $L^2(0,1\ )$Then $T_n$fails to be strongly compact thanks to Corollary2. Notice that the radial segments I_n. are pairwise disjoint and therefore the measures $\mu_n$are pairwise orthogonal. Consider the measure $\mu = \sum_{n=1}^{\infty} \mu_n/2^n$ .let Tdenote the operator of multiplication by$\mathcal{Z}$on $L^2(\mu\ )$ and observe that, thanks to Lemma4, $T$and $\oplus_{n=1}^{\infty} T_n$ are unitarily equivalent. Finally, let us showthat $T$is strongly compact. Since the union of the radial segmentsI_n is a dense subset of $\mathbb{D}$ it follows that any bounded sequence of polynomials in $L^{\infty}(\mu\ )$ is uniformly bounded on $\mathbb{D}$, and by Montel's Theorem ithas a subsequence that converges uniformly on compact subsets of$\mathbb{D}$, so that condition (iii) of Theorem 1, is satisfied.

Now we provide an example of a strongly compact direct sum of two non strongly compact normal operators.

Proposition 6. *There exist two non strongly compact normal operators such that their direct sum is strongly compact.*

Proof. Let$D_1 = \{z\ \in \mathbb{C} : |\mathcal{Z} - 1/2| < 1\}$ and $D_2\ = \{z\ \in \mathbb{C} : |z + 1/2| < 1\}$, let $\sigma_1$denote the two-dimensional Lebesgue measure on D_1, and let $\sigma_2$denote the two-dimensional Lebesgue measure on$D_2 \backslash D_1$. We will consider the arc-length measure on the boundary ∂D_j of D_j, more precisely τ_1, will bethe normalized arc-length measure on the arc $(\partial D_1) \cap D_2$and $\tau_2$will be the normalized arc-length measure on the arc$(\partial D_2) \cap D_1$. Finally, let $\mu_j = \sigma_j + \tau_j$and denote by $T_j$the operator of multiplication by$\mathcal{Z}$on $L^2(\mu_j)$. for $j = 1\ , 2$.

The four measures $\sigma_1\ , \sigma_2$, τ_1 and τ_2 are pairwise orthogonal, so that μ_1 and μ_2 are orthogonal, andtherefore $T_1 \oplus T_2$ is unitarily equivalent to the operator of multiplication by$\mathcal{Z}$on $L^2(\mu_1 + \mu_2)$. A sequence ofpolynomials which is bounded on $L^{\infty}(\mu_1 + \mu_2)$ is uniformly bounded on$D_1 \cup D_2$, and by Montel's Theoremit has a subsequence that converges pointwise on $D_1 \cup D_2$ and therefore$(\mu_1 + \mu_2)$-almost everywhere. It follows from Theorem 1, that$T_1 \oplus T_2$ is strongly compact.

In order to prove that T_1 is not strongly compact, observe that $(\partial D_1) \cap D_2$ is an arc whose length isexactly one third of the length of the circle $\{\mathcal{Z}\ \in \mathbb{C} : |\mathcal{Z} - 1/2| < 1\}$, and it is easy to check for $p_n(\mathcal{Z}) = (\mathcal{Z}\ - 1/)^{3n}$that the sequence of polynomials $(p_n)_n$ is an orthonormal system in $L^2(\ \tau_1)$. This sequence is bounded in $L^{\infty}(\mu_1)$, but it is not relatively compact in $L^2(\mu_1)$because it is not so in $L^2(\ \tau_1\ )$. This shows that T_2 is not strongly compact. A similar argument with the sequence of polynomials$p_n(\mathcal{Z}) = (\mathcal{Z}\ + 1/)^{3n}$shows that T_2 is not strongly compact.

Now we turn to the general case of a normal operator Non a separable Hilbert space H.We start with the following result.

Lemma 5. *Let $\Omega \subseteq \mathbb{C}$ be a compact set and let $(\mu_n)_{n\geq 1}$be a sequence of probability measures defined on the Borel subsets of Ω, let $T_n$denote the operator of multiplication by z on $L^2(\mu_n)$,and assume that everyμ_n is absolutely continuous with respect to μ_1. Then $T = \oplus_{n=1}^{\infty} T_n$is strongly compact if and only if $T_1$is strongly compact.*

Proof . Notice that for any polynomial $p$we have

$$\|p(T)\| = \sup_n \|p(T_n)\| = \sup_n \|p\|_{L^{\infty}(\mu_1)}$$
$$= \|p\|_{L^{\infty}(\mu_1)} = \|p(T_1)\|.$$

Assume $T$is strongly compact and notice that for every $f_1\ \in L^2(\mu_1)$the set $\{p(T_1)f_1 : \|p(T_1)\| \leq 1\}$is the projection onto the first coordinate of the set

$\{ p(T)(f_1, 0, 0, \dots) : \|p(T_1)\| \le 1\}$, which is relativelycompact, and therefore T_1 is strongly compact.

Conversely, assume T_1 is strongly compact and take a sequence of polynomials $(p_j)_j$ such that $\|p_j(T)\| \le 1$ for every $j \ge 1$. Since $\|p_j(T)\| = \|p_j(T_1)\|$ and T_1 is strongly compact, there is a subsequence $(p_{jk})_k$ that converges μ_1-almost everywhere. Since μ_n is absolutely continuous with respect to μ_1, it follows that $(p_{jk})_k$ converges μ_n-almost everywhere. Hence, for every $f_n \in L^2(\mu_n)$, the set $\{p(T_n)f_n : \|p(T)\| \le 1\}$ is relatively compact. Now fix $f \in H = \oplus_{n=1}^{\infty} L^2(\mu_n)$, let $\varepsilon > 0$, and choose n_0 so large that

$$\sum_{n=n_0+1}^{\infty} \|f_n\|^2 < \varepsilon^2$$

Consider the compact set $K_\varepsilon = \{(p(T_1)f_1, \dots, p(T_n)f_n, 0, 0, \dots) : \|p(T)\| \le 1\}$ we have

$$\{p(T)f : \|p(T)\| \le 1\} \subseteq K_\varepsilon + \varepsilon B_H$$

so it follows from Lemma 1 and Lemma 1 that T is strongly compact.

Now we characterize strong compactness for a normal operator on a separable Hilbert space, that is, for a direct sum of operators of multiplication by $\mathcal{Z}$. We will consider only countable direct sums, but a similar result holds for finite sums.

Theorem 2. *Let $\Omega \subseteq \mathbb{C}$ be a compact set, let $(\mu_n)_n$ be a sequence of probability measures defined on the Borel subsets of Ω, let T_n be the operator of multiplication by $\mathcal{Z}$ on $L^2(\mu_n)$, and let $\mu = \sum_{n=1}^{\infty} \mu_n/2^n$. Then $\oplus_{n=1}^{\infty} T_n$ is strongly compact if and only if the operator T of multiplication by $\mathcal{Z}$ on $L^2(\mu_n)$ is strongly compact.*

Proof. It is clear that μ_n is absolutely continuous with respect to μ. Let f_n denote the Radon-Nikodymderivative of μ_n with respect to μ and let $A_n = \{\mathcal{Z} \in \Omega : f_n(\mathcal{Z}) > 0\}$. Then $\mu(\Omega \setminus \cup_{n=1}^{\infty} A_n) = 0$. Now choose $B_n \subseteq A_n$ such that $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$ and $(B_n)_n$ are pairwise disjoint. Let $v_n = \mu_n | B_n$ and $\sigma_n = \mu_n | \Omega \setminus B_n$, so that $\mu_n = v_n + \sigma_n$. let S_n *and* R_n denote the operators of multiplication by $\mathcal{Z}$ on $L^2(v_n)$ and $L^2(\sigma_n)$, respectively. Also, let $v = \sum_{n=1}^{\infty} v_n/2^n$. We have

$$\bigoplus_{n=1}^{\infty} T_n = \bigoplus_{n=1}^{\infty} (S_n \oplus R_n) \cdot$$

Since the measures v_n are pairwise orthogonal, Lemma 4 gives that $\oplus_{n=1}^{\infty} L^2(v_n)$ is isometrically isomorphic to $L^2(v)$, and the operator $\oplus_{n=1}^{\infty} S_n$ corresponds with the operator S of multiplication by z on $L^2(v)$. Thus, $\oplus_{n=1}^{\infty} T_n$ corresponds with $S \oplus (\oplus_{n=1}^{\infty} R_n)$. Now every σ_n is absolutely continuous with respect to v and it follows from Theorem 5 that $S \oplus (\oplus_{n=1}^{\infty} R_n)$ is strongly compact if and only if S is strongly compact. Finally, μ and v are equivalent measures, and by Lemma 3, T is strongly compact if and only if so is S, and Theorem 2, follows.

Recall that a spectral measure on a Hilbert space H is any measure defined on the Borel subsets of a compact set $\Omega \subseteq \mathbb{C}$ with values on the orthogonal projections of H which is countably additive in the strong operator topology and such that $E(\Omega) = I$. A common version of the Spectral Theorem [5] ensures that if N is a normal operator on a Hilbert space H then there exists a unique spectral measure E on H defined on the Borel subsets of $\sigma(N)$ such that

$$N = \int_{\sigma(N)} \lambda dE_\lambda .$$

The following result ensures that for every spectral measure on a separable Hilbert space one can find a probability measure whose null sets are those of the given spectral measure.

Lemma 6. *Let N be a normal operator on a separable Hilbert space H and let E be the spectral measure associated to N. Then there exists a probability measure v such that $E(B) = 0$ if and only if $v(B) = 0$.*

Proof. Let $(x_n)_n$ be a dense sequence in B_H and define a sequence of finite positive measures $(V_n)_n$ on the Borel subsets of $\sigma(N)$ by $v_n(B) = (E(B)x_n, x_n)$. Then put $= \sum_{n=1}^{\infty} v_n/2^n$. It is clear that if $E(B) = 0$ then $V_n(B) = 0$ for every n. Conversely, if $v_n(B) = 0$ for every n then $(E(B)x_n, x_n) = 0$ forevery $x \in H$ and since $E(B)$ is an orthogonal projection it follows that $E(B) = 0$.

We finish this section with a characterization of strong compactness for normal operators on a separable Hilbert space in terms of their spectral measures.

Theorem. *Let E be the spectral measure associated to a normal operator N on a separable Hilbertspace and let v be a probability measure defined on the Borel subsets of $\sigma(N)$such that $E(B) = 0$ if andonly if $\sigma(N) = 0$. Then N is strongly compact on H if and only if the operator of multiplication by $\mathcal{Z}$isstrongly compact on $L^2(v)$.*

Proof. We know $N$is unitarily equivalent to a direct sum of multiplications by z. We will consider onlythe case of an infinite sum, the case of a finite sum being similar. Thus, there exists a sequence $(\mu_n)_n$ofprobability measures on $\sigma(N)$ such that N is unitarily equivalent to$\oplus_{n=1}^{\infty} T_n$, where $T_n$is the operator of multiplication by $\mathcal{Z}$on $L^2(\mu_n)$. Let $\mu = \sum_{n=1}^{\infty} \mu_n/2^n$. From Theorem 2we know that N is strongly compact if and only if so is the operator of multiplication by $\mathcal{Z}$on $L^2(\mu)$. If we prove that μ and v are equivalent, we will be done thanks to Lemma 3. The spectral measure of N is equivalent toE_1 the spectral measure of $\oplus_{n=1}^{\infty} T_n$. It is easy to check that for every Borel set $B \subseteq \sigma(N), (B)$is the projection defined by

$$E_1(B): \oplus_{n=1}^{\infty} L^2(\mu_n) \to \oplus_{n=1}^{\infty} L^2(\mu_n).$$
$$(f_n)_n \longmapsto (f_n \chi \mathrm{B})_n$$

and therefore

μ (B) = 0$\Leftrightarrow$ μ_n $(B)= 0$ for all $n \Leftrightarrow E_1(B) = 0$ $\Leftrightarrow E(B) = 0 \Leftrightarrow v(B) = 0$,

so that $\mu$and $v$are equivalent measures.

In case the Hilbert space $H$is not separable, it is not true that for any normal operator$N$there is always a probability v equivalent to the spectral measure Eassociated to N. In fact, it can be proven that such a probability exists if and only if the set $\{E_{x,x}: x \in H\}$is separable in the space $M(\sigma(N))$of Radon measures on $\sigma(N)$,where $E_{x,x}$is the measure defined by $E_{x,x}(B) = (E(B)_{x,x})$. However, if such aprobability exists then Theorem 3, still holds, the arguments being similar. For instance, Lemma 5, is easily seen to be true for an uncountable family of measures.

4. Strongly compact weighted shifts

Let H be an infinite-dimensional, separable Hilbert space, let $(e_n)_{n\geq 0}$, be an orthonormal basis for H, and consider the weighted shift Winduced on $H$by a bounded sequence $(\mathcal{W}_n)_n$ of non-zero weights, that is, $We_n = w_n e_{n+1}$for every $n \geq 0$.

The aim of this section is to discuss the conditions under which a weighted shift is strongly compact. We start off with a necessary condition. We will show at the end of this section that this condition is not sufficient.

Proposition. *Let W be the weighted shift induced by a bounded sequence $(\mathcal{W}_n)_{n\geq 0}$of non-zero weights. If W is strongly compact then*

$$\lim_{n\to\infty} \inf_{k\geq 0} \left| \frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{k+j}} \right| = 0.$$

Proof. It is not hard to prove and it is shown in Halmos's book [4] that the norm of $W^n, n \geq 1$, is thesupremum of the sliding products of length n, or, explicitly

$$\| W^n \| = \sup_{k\geq 0} \left| \prod_{j=0}^{n-1} \mathcal{W}_{k+j} \right|.$$

The sequence of operators $(W^n/\| W^n \|)_{n\geq 0}$ lies inside the unit ball of the algebra generated by $W$and I. Hence, the sequence of vectors $(W^n e_0/\| W^n \|)_{n\geq 0}$ is relatively compact. On the other hand

$$\frac{W^n e_0}{\| W^n \|} = \left(\frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{k+j}} \right) e_n \to 0 \text{weakly as } n \to \infty$$

It follows that $\| W^n e_0 \| / \| W^n \| \to 0$ as $\to \infty$, that is

$$\lim_{n\to\infty} \inf_{k\geq 0} \left| \frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{k+j}} \right| = 0.$$

The necessary condition stated in Proposition 7, can be used for constructing a counterexample to show that the condition of having dense range cannot be removed from Proposition 2.

Proposition 8. *There exists a compact operator with the property that none of the* *restrictions to its nonzero invariant subspaces is strongly compact.*

Proof. Let Wbe a weighted shift with decreasing sequence $(\mathcal{W}_n)_{n\geq 0}$ of positive weights such that $\mathcal{W}_n \to 0$as $\to \infty$. Then $W$is compact and

$$\inf_{k\geq 0}\left|\frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{k+j}}\right| = 1 \ ,$$

so that $W$is not strongly compact. A Theorem of Nikolskii's [6] ensures that if

$$\sum_{n=0}^{\infty} w_n^2 < \infty$$

then the non-trivial invariant subspaces of $W$are all of the form $E =$ span$\{e_{n_0}, e_{n_{0+1}}, \dots\}$for some $n_0 \geq 0$. But then again $W_{|E}$is a weighted shift with decreasing sequence $(\mathcal{W}_n)_{n\geq n_0}$of positive weights, so that $W_{|E}$is not strongly compact. A proof of Nikolskii's Theorem can be found in Halmos's book. [4]

The following result gives another example of a restriction of a strongly compact operator to an invariant subspace that is not strongly compact.

Proposition 9. *There exists a strongly compact bilateral weighted shift with an invariant subspace such that the restriction of the operator to the invariant subspace fails to be strongly compact.*

Proof. Let$(\mathcal{W}_n)_{n\geq 0}$ be a decreasing sequence of positive weights such that $\mathcal{W}_n \to 0$ as $n \to \infty$, and put$w_{-n} = w_n$. Now let W be the bilateral weighted shift with sequence of weights $(w_n)_{n\in\mathbb{Z}}$. Then $W$is a compact operator with dense range, so by Proposition 2, we know that $W$is strongly compact. Let$E$ =span $\{e_0, e_1, \dots\}$. Then E is invariant under $W$and it follows from Proposition 8, that $W_{|E}$fails to be strongly compact, as we wanted.

Now we turn to the search of sufficient conditions for a weighted shift to be strongly compact.

Theorem 4. *Let W be the weighted shift induced by a bounded sequence $(\mathcal{W}_n)_{n\geq 0}$of non-zero weights and assume that*

$$\sum_{n=0}^{\infty} \inf_{k\geq 0}\left|\frac{\prod_{j=0}^{n-1} \mathcal{W}_j}{\prod_{j=0}^{n-1} \mathcal{W}_{k+j}}\right| < \infty$$

Then W is strongly compact.

Proof. It follows from the computation in the proof of Proposition 7, that

$$\frac{\| W^n e_0\|}{\| W^n\|} = \inf_{k\geq 0}\left|\frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{j+k}}\right| ,$$

so that the assumption is

$\sum_{n=1}^{\infty} \frac{\| W^n e_0\|^2}{\| W^n\|^2} < \infty$.

Since e_0 is a cyclic vector for, thanks to the remarks after Lemma 2, it suffices to show that

$S = \{ p(W)_{e_0}: p$ is a polynomial, $\|p(W)\| \leq 1\}$

is a relatively compact subset of H . Fix $\varepsilon>0$ and choose n_0 so large that

$\sum_{n=n_0}^{\infty} \frac{\| W^n e_0\|^2}{\| W^n\|^2} < \varepsilon^2.$

Let $K_\varepsilon$denote the unit ball of span $\{e_0, e_1, \dots, e_{n_0}\}$and let $p(z) = a_0 + a_0 z$ $+\dots+a_N z^N$be any polynomial such that $\|p(W)\| \leq 1$. We have for every $k \geq 0$

$p(W)e_k = a_0 e_k + \sum_{n=1}^{N}\left(a_n \prod_{j=0}^{n-1} w_{k+j}\right)e_{k+n}$,

So that

$|a_0|^2 + \sum_{n=1}^{N}\left(|a_n|^2\left|\prod_{j=0}^{n-1} w_{k+j}\right|^2\right) = \|p(W)e_k\|^2 \leq 1$ (*)

In particular, with $k = 0$ we get

$$\left\|\sum_{n=0}^{n_0} a_n W^n e_0\right\|^2 = |a_0|^2 \sum_{n=1}^{n_{0-1}}\left(|a_n|^2\left|\prod_{j=0}^{n-1} w_j\right|^2\right) \leq |a_0|^2 + \sum_{n=1}^{N}\left(|a_n|^2\left|\prod_{j=0}^{n-1} w_j\right|^2\right) = \|p(W)e_k\|^2 \leq 1.$$

On the other hand, it follows from (*) that for every n with $1 \leq n \leq N$we have

$|a_n|^2\left|\prod_{j=0}^{n-1} w_{k+j}\right|^2 \leq 1,$

and taking the supremum over $k \geq 0$ yields$|a_n|^2\| W^n\|^2 \leq 1$. Thus,

$$\left\|\sum_{n=n_0}^{N} a_n W^n e_0\right\|^2 = \sum_{n=n_0}^{N}\left(|a_n|^2\left|\prod_{j=0}^{n-1} w_j\right|^2\right) = \sum_{n=n_0}^{N}|a_n|^2\| W^n e_0\|^2 = \sum_{n=n_0}^{\infty}|a_0|^2 \frac{\| W^n e_0\|^2}{\| W^n\|^2}\| W^n\|^2 \leq \sum_{n=n_0}^{\infty} \frac{\| W^n e_0\|^2}{\| W^n\|^2} < \varepsilon^2$$

Hence, $(W)e_0 \in K_\varepsilon + \varepsilon BH$, and since $\varepsilon > 0$ is arbitrarily small, it follows from Lemma 1, that S is a relatively compact subset of H.
It will be shown in Corollary 4, below that the sufficient condition stated in Theorem 4, is not necessary for a weighted shift to be strongly compact. Anyway, in the next corollary we apply this sufficient condition to give an example of a compact weighted shift with non-zero weights which is strongly compact.

Corollary 3. *Let (ε_h) be a sequence of positive numbers such that*

$$\varepsilon_h \le \frac{1}{2^{h/2} h^{h+1}}$$

for every $h \ge 1$, and define (w_n) by

$$w_n = \begin{cases} 1, & if\ n = 0, 1 \\ \varepsilon_h, & if\ n = 2^h \\ \frac{1}{h}, & if\ 2^h < n < 2^{h+1} \end{cases}$$

Then the weighted shift W induced by the sequence of weights (w_n) is strongly compact.

Proof. Put

$$a_n = \inf_{k\ge 0} \left| \frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{k+j}} \right|,$$

take an $n \ge 2$, and choose $h \ge 1$ such that $2^h \le n < 2^{h+1}$. Notice that between 2^h and 2^{h+1} there are at least h integers, so that $a_n \le h^h \varepsilon h$. Thus,

$$\sum_{n=1}^{\infty} a_n^2 \le 1 + \sum_{n=1}^{\infty} \sum_{n=2^h}^{2^{h+1}-1} a_n^2 \le 1 + \sum_{h=1}^{\infty} 2^h h^{2h} \varepsilon_h^2 \le 1 + \sum_{h=1}^{\infty} \frac{1}{h^2} < \infty .$$

and it follows from Proposition 4, that W is strongly compact.

Now look at the condition stated in Proposition 7, and assume that the sequence of weights has increasing moduli, that is, $|w_n| \le |w_{n+1}|$ for all $n \ge 1$. Then there exists $\lambda = \lim |w_n|$ and therefore

$$\lim_{n\to\infty} \inf_{k\ge 0} \left| \frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{k+j}} \right| = \lim_{n\to\infty} \frac{\left|\prod_{j=0}^{n-1} w_j\right|}{\lambda^n}$$

It turns out that under the monotony assumption the necessary condition stated in Proposition 7, is also sufficient for a weighted shift to be strongly compact.

Theorem 5. *Let W be the weighted shift induced by a bounded sequence $(w_n)_{n\ge 0}$ of non-zero weights with increasing moduli and assume that*

$$\lim_{n\to\infty} \inf_{k\ge 0} \left| \frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{k+j}} \right| = 0.$$

Then W is strongly compact.

Proof. Since e_0 is a cyclic vector for W, thanks to the remarks after Lemma 2, it suffices to show that the set

$S = \{ : p(W)\, e_0$ is a polynomial, $\|p(W)\| \le 1\}$

is relatively compact in H . Let $\lambda = \lim |w_n|$, fix $\varepsilon > 0$, and choose n_0 so large that if $n \ge n_0$ then

$\left|\prod_{j=0}^{n-1} w_j\right| < \varepsilon \lambda^n$.

Let K_ε denote the unit ball of span$\{e_0, e_1, \dots, e_{n_0}\}$ and let $p(\mathcal{Z}) = a_0 + a_0\mathcal{Z} + \dots + a_N \mathcal{Z}^N$ be any polynomial such that $\|p(W)\| \le 1$. We have for every $k \ge 0$.

$p(W)\, e_0 = a_0 e_k + \sum_{n=1}^{N} \left(a_n \prod_{j=0}^{n-1} w_{k+j}\right) e_{k+n}$,

so that

$|a_0|^2 + \sum_{n=1}^{N} \left(|a_n|^2 \left|\prod_{j=0}^{n-1} w_{k+j}\right|^2\right) = \|p(W) e_k\|^2 \le 1$ (*)

In particular, with $k = 0$ we get

$$\left\|\sum_{n=0}^{n_0} a_n W^n e_0\right\|^2 = |a_0|^2 + \sum_{n=1}^{n_{0-1}} \left(|a_n|^2 \left|\prod_{j=0}^{n-1} w_j\right|^2\right) \le$$

$$|a_0|^2 + \sum_{n=1}^{N} \left(|a_n|^2 \left|\prod_{j=0}^{n-1} w_j\right|^2\right) = \|p(W) e_k\|^2 \le 1$$

On the other hand, taking limits as $k \to \infty$ in (*) gives

$$|a_0|^2 + \sum_{n=1}^{N} |a_n|^2 \lambda^{2n} \le 1 ,$$

So that

$$\left\|\sum_{n=n_0}^{n_0} a_n W^n e_0\right\|^2 = \sum_{n=n_0}^{N}\left(|a_n|^2\left|\prod_{j=0}^{n-1} w_j\right|^2\right) \le \mathcal{E}^2 \sum_{n=n_0}^{N}|a_n|^2\lambda^{2n} \le \mathcal{E}^2$$

Hence, $p(W)\, e_0 \in K_\varepsilon + \varepsilon B_H$, and since $\varepsilon > 0$ is arbitrarily small, it follows from Lemma 1, that S is a relatively compact subset of H.

Observe that when $(|w_n|)$ is an increasing sequence that converges to, the condition stated in Theorem 5, is just

$\lim_{n\to\infty} \prod_{j=0}^{n-1} \frac{|w_j|}{\lambda} = 0$,

and this happens if and only if

$$\sum_{n=1}^{\infty}(\lambda - |w_n|) = \infty$$

As an application of Theorem 5, we come back to Corollary 1and take a look at the operator of multiplication by $\mathcal{Z}$on Bergman space from a different point of view.

Corollary 4. *The operator $M_\mathcal{Z}$of multiplication by $\mathcal{Z}$ on $A^2(\mathbb{D})$ is strongly compact.*

Proof. It is shown in Halmos's book [4] that $A^2(\mathbb{D})$ is isometrically isomorphic to a weighted sequence space so that $M_\mathcal{Z}$is unitarily equivalent to the weighted shift Wwhose weights are given by

$$w_n = \left(\frac{n+1}{n+2}\right)^{1/2}$$

A quick computation shows that this sequence of weights satisfies the condition of Theorem 5, Thus, $M_\mathcal{Z}$is strongly compact on $A^2(\mathbb{D})$.

Notice that this weighted shift does not satisfy the condition stated in Theorem 4, and therefore that condition is not necessary for a weighted shift to be strongly compact.

We finish this section with an example to show that the necessary condition stated in **Proposition 7,** in general is not sufficient for a weighted shift to be strongly compact.

Proposition 10.*There exists a sequence of non zero weights $(w_n)_{n\ge 0}$such that the corresponding weighted shift is not strongly compact but*

$$\lim_{n\to\infty} \inf_{k\ge 0}\left|\frac{\prod_{j=0}^{n-1} w_j}{\prod_{j=0}^{n-1} w_{k+j}}\right| = 0.$$

Proof. Let $(n_k)_{k\ge 1}$ be an increasing sequence of positive integers satisfying

(i) $n_{k+1} > 4n_k$.

(ii)$(k+2)^{n_k/n_{k+1}} \le 2.$

For instance, the sequence $n_k= (k+3)!$ works. Let$\rho_k = (k+1)^{1/n_k}$.Then $(\rho_k)_k$is a decreasing sequence in the interval, [1, 2] since $n_k \le (k+1)^{1/k} \le 2$ and, thanks to (i),

$$\rho_k^{n_k} = k+1 \ge (k+2)^{1/4} > (k+2)^{n_k/n_{k+1}} = \rho_{k+1}^{n_k}.$$

Define the sequence $(w_n)_{n\ge 0}$ of positive weights by $w_n= 1$ for $n \le n_1$, and if $n_k < n \le n_k$+1 for certain k then

$$w_n = \begin{cases} 1/\sqrt{k+1} \quad , if\ n = n_{k+1} - 2n_k\ or\ n = n_{k+1} \\ 1\ , if\ n_k < n < n_{k+1} - 2n_k\ or\ n_{k+1} - n_k < n < n_{k+1}\ , \\ \rho_k, if\ n_{k+1} - 2n_k < n \le n_{k+1} - n_k\ . \end{cases}$$

Let W denote the corresponding weighted shift. We have $W^n e_0= \tau_n e_n$, where $\tau_n=\prod_{j=0}^{n-1} \mathcal{W}_j$. It is easy to check that $\tau_n = 1$for $\le n_1$, and for every k ≥ 1

$$\tau_n = \begin{cases} 1\ , \quad if\ n_k < n \le n_{k+1} - 2n_k\ , \\ \rho_k^{\,n-(n_{k+1}-2n_k+1)}/\sqrt{k+1} \quad , if\ n_{k+1} - 2n_k < n \le n_{k+1} - n_k \\ \sqrt{k+1} \quad , if\ n_{k+1} - n_k < n \le n_{k+1}. \end{cases}$$

Observe that for $n \le n_{k+1}$ we always have $1/\sqrt{k+1} \le \tau_n \le \sqrt{k+1}$. In order to prove that $\| W^n e_0\| / \| W^n\| \to 0\ as\ n \to \infty$. we are going to distinguish two cases. First of all, if $n_k < n \le n_{k+1}$for some k≥ 1then $\| W^n e_0\|= \tau_n= 1$, and

$$\| W^n\| \ge \| W^n e_{n_{k+2}-2n_{k+1}}\| = \prod_{m=n_{k+1}-2n_{k+1}+1}^{n_{k+1}-2n_{k+n}} w_m$$

$$= \frac{t_{n_{k+1}-2n_{k+n+1}}}{t_{n_{k+1}-2n_{k+1}}} \ge \frac{1}{1/\sqrt{k+1}} = \sqrt{k+1}\,,$$

using that, thanks to (i), $n_{k+1} - n_k < n_{k+1} - 2n_{k+n+1} < 2n_{k+1} < n_{k+2} - 2n_{k+1}$.

Therefore, in this case,$\| W^n e_0\| / \| W^n\| \le 1/\sqrt{k+1}$. Otherwise, if $n_{k+1} - 2n_k < n \le n_{k+1}$then $\| W^n e_0\|= \tau_n \le \sqrt{k+1}$, and using (ii) we get

$$\| W^n\| \ge \| W^n e_{n_{k+2}-2n_{k+1}}\| = \prod_{m=n_{k+2}-2n_{k+1}+1}^{n_{k+2}-2n_{k+n}} w_m = \rho_{k+1}^n$$

$$\ge \rho_{k+1}^{n_{k+1}-2n_k} = (k+2)(k+2)^{-2n_k/n_{k+1}} \ge \frac{k+2}{4}\,.$$

These estimates yield $\| W^n e_0\|/\| W^n\| \to 0\ as\ n \to \infty$. In order to check that W is not strongly compact we will consider the sequence of polynomials $(p_k)_k$ defined by

$p_k(\mathcal{Z}) = \frac{1}{6k}\sum_{i=k+1}^{2k} \mathcal{Z}^{n_i}$.

This sequence of polynomials satisfies the following property.

Claim A. For every $k \geq 1$ we have $\|p_k(w)\| \leq 1$.

Before proving Claim Awe finish the proof of Proposition 10. This is now easy, since

$$p_k(w)e_0 = \frac{1}{6k}\sum_{i=k+1}^{2k} \sqrt{i} e_{ni} \to 0\ weakly\ as k \to \infty\ ,$$

and

$$\|p_k(w)e_0\|^2 = \frac{1}{36k^2}\sum_{i=k+1}^{2k} i \geq \frac{1}{36}\frac{k(k+1)}{k^2} \geq \frac{1}{36},$$

so that the set $\{p(w)e_0 : \|p(w)\| \leq 1\}$is not relatively compact and therefore $W$is not a strongly compact operator.

Notice that for every $i \geq 1$and $j \geq 0$ there exists $a_{ij} > 0$such that $W^{ni}e_j = a_{ij}e_{j+ni}$.We will need the following estimates fora_{ij}

Claim B. For every $i \geq 1$ and $j \geq 0$we have

$$a_{ij} \leq \begin{cases} i+1, & \sqrt{i},\ if\ j < n_i - n_{i-1} \\ & if\ n_i - n_{i-1} \leq j < n_{i+1} - n_i \\ & 2,\ if\ n_{i+1} - n_i \leq j \end{cases}$$

Proof of Claim B. We know that$a_{ij} = \prod_{k=j}^{j+n_i-1} w_k = \tau_{j+n_i}/\tau_j$. $if\ j = 0$ then$a_{ij} = \tau_{n_i} = \sqrt{i}$, and if $0 < j < n_i - n_{i-1}$ then $n_i < j + n_i < 2n_i < n_{i+1} - 2n_i$, so that$\tau_{j+n_i} = 1\tau_j \geq 1/\sqrt{i}$. This takes care of the first estimate. Now observe that $1/\sqrt{k+1} \leq \tau_n \leq \sqrt{i+1}$ for every $n \leq n_{i+1}$, and $a_{ij} = \tau_{j+n_i}/\tau_j \leq i+1$, since $j, j+n_i \leq n_{i+1}$if $j< n_{i+1} - n_i$Thus, the second estimate follows. Inorder to get the third estimate, notice that for $j = n_{i+1} - n_i$we have $a_{ij} = \rho_i \leq 2$ and for $j > n_{i+1} - n_i$, we have $w_j \leq \rho_{i+1}$and so $a_{ij} =\leq \rho_{i+1}^{n_i} = (i+2)^{\frac{n\ i}{n_{i+1}}} \leq 2$, thanks to (ii), and the proof of Claim Bis finished.

Proof of Claim A. Fix an $i$such that $k + 1 \leq i \leq 2k$and let us split W^{n_i} as the sum of two operators $W^{n_i} = S_i + T_i$defined by

$$S_i e_j = \begin{cases} W^{n_{ie_j}}, if\ j = n_{i+1} - n_i \\ 0,\ if\ j \geq n_{i+1} - n_i \end{cases}$$

and

$$T_i e_j = \begin{cases} 0,\ if\ j > n_{i+1} - n_i \\ W^{n_{ie_j}}, if\ j \geq n_{i+1} - n_i \end{cases}$$

The third estimate of Claim B. yields $\|T_i\| \leq 2$for $k + 1 \leq i \leq 2k$ and so

$\left\|\frac{1}{6k}\sum_{i=k+1}^{2k} T_i\right\| \leq \frac{1}{6k}\sum_{i=k+1}^{2k}\|T_i\| \leq \frac{1}{3}$.

Now let $R = \sum_{i=k+1}^{2k} S_i$.Our aim is to show that $\|R\| \leq 4k$because this will produce

$\|p_k(w)\| \leq \frac{1}{6k}\|R\| + \left\|\frac{1}{6k}\sum_{i=k+1}^{2k} T_i\right\| \leq \frac{2}{3} + \frac{1}{3} = 1$.

which is what we want.

In order to estimate the norm of R we observe that if $S_i e_j \neq 0$ then $j< n_{i+1} - n_i$ and $n_i \leq j + n_i < n_{i+1}$. This implies that if $i \neq i'$then $S_i e_j$and $S_i e_{j'}$ are orthogonal vectors, no matter what $j$and $j'$are. On the other hand, if $j \neq j'$then $S_i e_j$and $S_i e_{j'}$are also orthogonal. Therefore $Re_j$is orthogonal to $Re_{j'}$ whenever $j \neq j'$and we may conclude that

$\|R\| = \sup_{j \geq 0}\|Re_j\|$.

Now define $I_j = \min\{i \geq 1 : n_{i+1} - n_i > j\}$and notice that

$$Re_j = \sum_{k+1 \leq i \leq 2k. i \geq I_j} a_{ij} e_{j+n_i}$$

$If I_j > 2k then \|Re_j\| = 0$ Assume $I_j \leq 2k$. Using the estimates of B and observing that

$\|Re_j\|^2 = \sum_{k+1 \leq i \leq 2k. i \geq I_j} a_{ij}^2 \leq a^2{}_{I_j j}$

$$\leq (I_j + 1)^2 + \sum_{i=k+1}^{2k} i \leq (2k+1)^2 + 2k^2 \leq 16k^2 .$$

Thus, $\|Re_j\| \leq 4k\ for\ every\ j \geq 0$, $\|R\| \leq 4k$ and Claim A follows.

Corollary 5. *Let* $\varepsilon(1+\theta)$ *be a sequence of positive numbers such that*

$$\varepsilon(1+\theta) \leq \frac{1}{2^{(1+\theta)}/_2(1+\theta)^{(\theta+2)}}$$

for $\theta > 0$, and define (W_n) by

$$w_n = \begin{cases} 1\ , if\ n = 0, 1 \\ \varepsilon(1+\theta), if\ n = 2^{(1+\theta)} \\ \frac{1}{h}, if\ 2^{(1+\theta)} < n < 2^{(\theta+2)} \end{cases} .$$

Then the weighted shift Winduced by the sequence of weights (W_n) is strongly compact. [7]

Proof . Put

$$\alpha_{(\theta+2)} = \inf_{k\geq 0}\left|\frac{\prod_{j=0}^{(1+\theta)} w_j}{\prod_{j=0}^{(\theta+2)} w_{k+j}}\right|$$

take an $\theta > 0$, and choose $\theta > 0$ such that $2^{(1+\theta)} < n < 2^{(\theta+2)}$. Notice that between $2^{(1+\theta)}$ and $2^{(\theta+2)}$ there are at least $(1+\theta)$ integers, so that$\alpha_{(\theta+2)} \leq (1+\theta)^{(1+\theta)}\varepsilon(1+\theta)$. Thus,

$$\sum_{\theta=1}^{\infty} \alpha_{(\theta+2)}^2 \leq 1 + \sum_{\theta=0}^{\infty}\sum_{n=2^{(1+\theta)}}^{2^{(\theta+2)}-1} \alpha_{(\theta+2)}^2 \leq 1 + \sum_{\theta=0}^{\infty} 2^{(\theta+2)(1+\theta)^{2(1+\theta)}}\varepsilon_{(1+\theta)}^2 \leq$$

$$1 + \sum_{\theta=0}^{\infty}\frac{1}{(1+\theta)^2} < \infty$$

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