International Journal of Research and Review
www.ijrrjournal.com E-ISSN: 2349-9788; P-ISSN: 2454-2237

Original Research Article

Solutions of the Hua System on Hermitian Symmetric Spaces of Tube Type

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ABSTRACT

In this paper, we give an over view results on the eigen functions of the Hua operator on a Hermitian symmetric space of tube type $X = G/K$, , let $\lambda_j \in \mathbb{C}$ $(j = 1,2, \dots\dots, n)$such that $\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$, and let F be a $\mathbb{C}$-valued function on X satisfying the following Hua system:

$$H_q F = \sum_{j=1}^{n} \frac{(\lambda_j^2 - \eta 2)}{32\eta 2} F Z .$$

Then F has an L^p-Poisson integral representation $(1 < p < +\infty)$ over the Shilov boundary of X .

Key words: Hua operator, tube type, eigenfunctions, Hua system,

INTRODUCTION

Let $X = G/K$ be an irreducible bounded symmetric domain of tube type and let $H_q$be the associated Hua operator on it. Shimeno [11] proved that the Poisson transform maps the space $\sum_{j=1}^{n} B(G/P_{\Xi};\lambda_j)$of hyperfunctions-valued sections of a degenerate principal series attached to the Shilov boundary $G/P_{\Xi}$of X, bijectively onto an eigenspace $\sum_{j=1}^{n} E_{\lambda_j}(X)$of the Hua operator H_q, under certain condition on the complex parameter $\lambda_1, \lambda_2, \dots \lambda_j$. Here $P_{\Xi}$is a certain maximal parabolic subgroup of G.
Now let $r$and m'denote respectively the rank of $X$and the multiplicity of the shortrestricted roots.

Theorem (1): [11] *Let $\lambda_1, \lambda_2, \dots \lambda_j$ be a complex number such that*

$$\sum_{j=1}^{n} -\lambda_j - \frac{m'}{2}(-r + 2 + i) \notin \{1, 2, 3, \dots\} for\ i = 0\ and\ 1. (1)$$

Then the Poisson transform is a G-isomorphism from $\sum_{j=1}^{n} B(G/P_{\Xi};\lambda_j)$ onto the space $\sum_{j=1}^{n} E_{\lambda_j}(X)$ of all analytic functions F on X that satisfy

$$H_q F = \sum_{j=1}^{n} \frac{\left(\lambda_j^2 - \left(1 + m'(r-1)/2\right)^2\right)}{32(1 + m'(r-1)/2)^2} FZ . (2)$$

Z being some specific element in the center of the Lie algebra of K.

In the light of the above result, it is natural to look for a characterization of the range of the Poisson transform on classical spaces on the Shilov boundary G/P_{Ξ}such as $C^{\infty}(G/P_{\Xi}), L^p(G/P_{\Xi})$and the space of distribution $D'(G/P_{\Xi})$.
In the case $\sum_{j=1}^{n} \lambda_j = 1 + m(r-1)/2$, i.e. the eigenspace consists of Hua harmonic functions,Koranyi and Malliavin [8] showed in the case of $X = Sp(2, \mathbb{R})/U(2)$that the image of the space$L^{\infty}(G/P_{\Xi})$is the space of all bounded Hua harmonic functions on X.

Actually, this result remains true for all Hermitian symmetric spaces of tube type, see. [5]

If $X = SU(n,n)/S(U(n)\times U(n))$, we showed in [2] that the Poisson transform attached to certain degenerate principal series of $SU(n,n)$is a topological isomorphism from $L^2(G/P_\Xi)$ onto a specific Hardy type space for eigenfunctions of the Hua operator on X.

The aim of this section is, on one hand, to extend in a unified manner the result in [2] to all Hermitian symmetric spaces of tube type and on the other hand, to characterize, for all $p, 1<p<+\infty$, the L^p-range of the Poisson transform in X.

Let $P_\Xi$be a maximal standard parabolic subgroup of $G$with Langlands decomposition $P_\Xi = M_\Xi A_\Xi N_\Xi$such that $A_\Xi$is of real dimension one. Then, the group Ghas the following generalized Iwasawa decomposition:

$$G = KM_\Xi A_\Xi N_\Xi^+.$$

For $x\in G$, we denote by $H_\Xi(x)$the unique element in a_Ξ($a_\Xi$being the Lie algebra of A_Ξ), such that $x \in KM_\Xi e^{H_\Xi(x)} N_\Xi^+$.

On the one-dimensional Lie algebra $a_\Xi = \mathbb{R}X_0$ we define the linear forms

$$\rho_0(X_0) = r \text{ and} \rho_\Xi = \left(1+\frac{m'}{2}(r-1)\right)\rho_0.$$

For $\lambda_1,\lambda_2,\dots\lambda_j\in\mathbb{C}$ we denote by $\sum_{j=1}^n B\big(G/P_\Xi;\ \lambda_j\big)$ the space of all hyper functions $f$on Gthat satisfy

$$f(gman) = \prod_{j=1}^n e^{(\lambda_j\rho_0-\rho_\Xi)H_\Xi(a)} f(g),\forall g\in G, m\in M_\Xi, a\in A_\Xi, n\in N_\Xi^+.$$

Then the Poisson transform $P_\lambda$of an element $f\in \sum_{j=1}^n B\big(G/P_\Xi;\ \lambda_j\big)$ is defined by

$$\sum\nolimits_{j=1}^n P_{\lambda_j} f(gK) = \int_K f(gk)\,dk\,.$$

A straightforward computation shows that

$$\sum\nolimits_{j=1}^n P_{\lambda_j} f(gK) = \int_K \prod_{j=1}^n e^{-(\lambda_j\rho_0+\rho_\Xi)H_\Xi(g^{-1}k)} f(k)dk\,.$$

By the decomposition $G = KP_\Xi$, the restriction from $G$to $K_\Xi = K\cap M_\Xi$gives a G-isomorphism from $\sum_{j=1}^n B\big(G/P_\Xi;\ \lambda_j\big)$ onto the space $B(K/K_\Xi)$of all hyperfunctions $f$on Kthat satisfy

$$f(km) = f(k), \forall m\in K_\Xi.$$

The classical space $L^p(K/K_\Xi)$will be regarded as the space of all $\mathbb{C}$-valued measurable (classes) functions $f$on Kwhich are right K_Ξ-invariant with $\|f\|_p < +\infty$. Here

$$\|f\|_p = \left[\int_K |f(k)|dk\right]^{1/p},$$

dkbeing the normalized Haar measure of the compact group K. Since the space $L^p(K/K_\Xi)$can be seen as a G-invariant subspace of $\sum_{j=1}^n B\big(G/P_\Xi;\ \lambda_j\big)$, then theimage $\sum_{j=1}^n P_{\lambda_j}\left(L^p(K/K_\Xi)\right)$is a proper closed subspace of $\sum_{j=1}^n E_{\lambda_j}(X)$, by Shimeno result, provided thatthe parameter $\lambda_1,\lambda_2,\dots\lambda_j\in\mathbb{C}$ satisfies (1).

Now, in order to characterize those F in $\sum_{j=1}^n E_{\lambda_j}(X)$that are Poisson transforms by $\sum_{j=1}^n P_{\lambda_j}$of some $f\in L^p(K/K_\Xi)$, we introduce a Hardy type space $\sum_{j=1}^n E^*_{\lambda_j,p}(X)$ for eigenfunctions of the Hua operator H_q.

More precisely, for $\lambda_1,\lambda_2,\dots\lambda_j\in\mathbb{C}$ and $p, 1<p<+\infty$, we define $\sum_{j=1}^n E^*_{\lambda_j,p}(X)$ to be the space of all functions F in $\sum_{j=1}^n E_{\lambda_j}(X)$ that satisfy

$$\sum_{j=1}^n \|F\|_{\lambda_j,p} = \sup_{a\in A_\Xi^+}\prod_{j=1}^n e^{(\rho_\Xi-\Re(\lambda_j)\rho_0)H_\Xi(a)}\left(\int_K |F(ka)|^p dk\right)^{1/p} < +\infty.$$

From now on, we will use on $A_\Xi$the coordinate $a_t = e^{tX_0}$; $t\in\mathbb{R}$. Henceforth the above norm becomes

$$\sum_{j=1}^n \|F\|_{\lambda_j,p} = \sup_{t>0}\prod_{j=1}^n e^{r(\eta-\Re(\lambda_j))t}\left(\int_K |F(ka_t)|^p dk\right)^{1/p},$$

where $\eta = 1+m'(r-1)/2$.

Notice that $2\eta$is the so called genus of the bounded symmetric domain X.

Also, we introduce a c-function given by the following integral representation:

$$\sum_{j=1}^n c(\lambda_j) = \int_{N_\Xi^-}\prod_{j=1}^n e^{-(\lambda_j\rho_0+\rho_\Xi)H_\Xi(n)}\,dn\,.$$

The above integral converges absolutely if and only if $\sum_{j=1}^n \Re(\lambda_j) > \eta-1$ (see Lemma (3)).

Notice that the introduced c-function $\sum_{j=1}^n c(\lambda_j)$ appears naturally in the study of the intertwining operators associated to the noncompact realization of the degenerate principal series representation of G. We have the following consequences

(i) As an immediate consequence of Theorem (11), we get that for $\lambda_1, \lambda_2, \dots \lambda_j \in \mathbb{C}$ such that $\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$ and for $p, 1 < p < +\infty$, the introduced Hardy type spaces $\sum_{j=1}^{n} E_{\lambda_j,p}^{*}(X)$are Banach spaces. This is closely-similar to O. Bray conjecture in the case of the real hyperbolic space with $p = 2$ (see [1,12]).

(ii) Letting $\sum_{j=1}^{n} \lambda_j = \eta$ in Theorem (11), we get a characterization of those Hua harmonic functions on X(i.e. $H_q F = 0$) that have an L^p-Poisson integral representations over the Shilov boundary $K/K_{\Xi}$of X. More precisely, let $H^p(X)$denote the space of all $\mathbb{C}$-valued Hua harmonic functions Fon Xsuch that

$$\sup_{t>0} \left(\int_K |F(ka_t)|^p dk \right)^{1/p} < +\infty.$$

Then we have

Corollary (2): *For $1 < p < \infty$, the Poisson transform P_η is a topological isomorphism from $L^p(K/K_{\Xi})$ onto $H^p(X)$.*

(iii) Since holomorphic functions on $X$are annihilated by the Hua operator we can use the above corollary to show that every holomorphic function on $X$with finite Hardy norm (i.e. $\sup_{t>0} \left(\int_K |F(ka_t)|^p dk\right)^{1/p} < +\infty$) has an L^p-Poisson integral representation over the Shilov boundary of X. Such result was earlier established by Koranyi using a different method (see [12,8,6]).

We are concerned on the minimal Hua system $H_q$introduced by Lassalle. [9] Since we will not require the general properties of $H_q$we will not need to recall its definition referring to Lassalle, [9] Helgason [5] (see also Faraut and Koranyi [3] for Jordan algebra theoretical setting of H_q).

In this section we recall some structural results on Hermitian symmetric spaces of tube type from [11] without proof.

For a real Lie algebra $b$we denote by $b_c$its complexification. Let Gbe a connected simple Lie group with finite center and let Kbe a maximal compact subgroup of G. Suppose that G/K is a Hermitian symmetric space of tube type and of rank $r > 1$.

Let $g = t \oplus p$be the Cartan decomposition of the Lie algebra $g$of $G$with Cartan involution θ. The center $z$of $t$is of dimension one and there exists $Z \in z$such that $(adZ)^2 = -1$ on p_c. Then $p_c$decomposes as $p^+ \oplus p^-$, the eigenspaces of $\pm i$, respectively.

Let h be a Cartan subalgebra of t(hence also of g). We denote by $\Delta$the set of roots of the pair (g_c, h_c). For $\gamma \in \Delta$, let $g_\gamma \subset g_c$be the root space for γ. We choose a set of positive roots Δ^+such that $p^+ = \sum_{\gamma \in \Delta_n^+} g_\gamma$, where $\Delta_n^+ = \Delta^+ \cap \Delta_n, \Delta_n$being the set of noncompact roots.

Let Bdenote the Killing form of g_c. For each $\gamma \in \Delta$we choose vectors $H_\gamma \in h_c, E_\gamma$and $E_{-\gamma}$such that $\overline{E_\gamma} = -E_{-\gamma}, [E_\gamma, E_{-\gamma}] = H_\gamma$, where the bar denotes the conjugation with respect tothe real form $t + ip$of g_c.

For $\alpha \in \Delta_n^+$, we set $X_\alpha = E_\alpha + E_{-\alpha}$and $Y_\alpha = i(E_\alpha + E_{-\alpha})$. Then it is well known that $X_\alpha$and $Y_\alpha$form a basis of p.

Let $\{\gamma_1, \dots, \gamma_r\}$be a maximal set of strongly orthogonal noncompact roots such that $\gamma_j$is the highest element of Δ_n^+strongly orthogonal to $\gamma_{j+1}, \dots, \gamma_r$for $j = r, \dots, 1$. Then the space $a = \sum_{j=1}^{r} \mathbb{R} X_{\gamma j}$is a maximal Abelian subspace of p.

Let Σ(respectively Σ^+) denote the set of restricted roots of the pair (g, a)(respectivelypositive roots in Σ). Then by classification, $\Sigma^+$is of the form

$$\Sigma^+ = \left\{\beta_i, \frac{\beta_j \pm \beta_k}{2}, 1 \le i \le r, 1 \le k < j \le r\right\},$$

where $\beta_i = \gamma_i \circ (c^{-1})|_a$, c being the Cayley transform of g_c.

We set

$$\alpha_j = \frac{\beta_{r-j+1} - \beta_{r-j}}{2}$$

for $1 \le j \le r - 1$ and $\alpha_r = \beta_1$.

Then $\Gamma = \{\alpha_1, \dots, \alpha_r\}$, is the set of simple roots in Σ^+.

For $\alpha \in \Sigma$let $g^\alpha \subset g$be the root space for $\alpha$and $m_\alpha$its multiplicity. As usual put $\rho = (1/2) \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. The multiplicity of long roots $m_{\beta_i}$for $i = 1, 2, \ldots, r$equals 1. We set $m' = m_{1/2\pm\beta_j\pm\beta_k}$for $1 \le j \ne k \le r$. For $\sum_{j=1}^{n} \lambda_j \in a_c^\star$, we denote by $\sum_{j=1}^{n} H_{\lambda_j}$the unique element in $a_C$such that $\sum_{j=1}^{n} B\left(H, H_{\lambda_j}\right) = \sum_{j=1}^{n} H_{\lambda_j}$ for all $H \in a$.

For $\lambda_1, \lambda_2, \ldots \lambda_j, \mu \in a_c^\star$we put $\sum_{j=1}^{n}\langle \lambda_j, \mu\rangle = \sum_{j=1}^{n} B\left(H_{\lambda_j}, H_\mu\right)$. Let W be the Weyl group of the pair (g, a). Then$W$acts on $a$and $a^\star$(via the Killing form) and is naturally identified with the Weyl group of Σ.

Let $\Xi = \Gamma \backslash \{\alpha_r\}$and let $P_\Xi$be the corresponding standard parabolic subgroup with the Langlands decomposition $P_\Xi = M_\Xi A_\Xi N_\Xi^+$such that $A_\Xi \subset A$, where $A$is the analytic subgroup of $G$with Lie algebra a. Then, it is well known that $P_\Xi$is a maximal standard parabolic subgroup of $G$and $G/P_\Xi$is the Shilov boundary of X. Moreover, G/P_Ξcan be identified to the compact symmetric space K/K_Ξ. If $a_\Xi$denotes the Lie algebra of A_Ξ. Then

$$a_\Xi = \{H \in a;\ \gamma(h) = 0, \forall_\gamma \in \Xi\}.$$

Moreover, $a_\Xi = \mathbb{R}X_0$, where $X_0 = \sum_{j=1}^{r} X_{\gamma_j}$.

Let $a(\Xi)$denote the orthogonal complement of $a_\Xi$in $a$with respect to the Killing form of g. Let $\rho_\Xi$and $\rho_{a(\Xi)}$be the restrictions of $\rho$to $a_\Xi$and $a(\Xi)$, respectively. Then $\rho = \rho_\Xi + \rho_{a(\Xi)}$. Moreover,

$$\rho_\Xi = \frac{1}{2} \sum_{\alpha \in \Sigma^+ \backslash \langle \Xi \rangle} m_\alpha \alpha\,,$$

and

$$\rho_a(\Xi) = \frac{1}{2} \sum_{\alpha \in \Sigma^+ \cap \langle \Xi \rangle} m_\alpha \alpha\,.$$

Here $\langle \Xi \rangle = \Sigma \cap \sum_{\alpha \in \Xi} \mathbb{Z}\alpha$.

Finally, we recall an integral formula on the group $N_\Xi^- = \theta(N_\Xi^+)$. Let dnbe the invariantmeasure on N_Ξ^-, normalized by$\int_{N_\Xi^-} e^{-2\rho_\Xi H_\Xi(n)} dn = 1$. Then, for a continuous function $f$onK/K_Ξwe have

$$\int_K f(k)dk = \int_{N_\Xi^-} f(\kappa(n)) e^{-2\rho_\Xi H_\Xi(n)} dn. \quad (3)$$

In the above formula $\kappa(x)$denotes the K-component of $x \in G$with respect to the decomposition $G = KM_\Xi A_\Xi N_\Xi$.

We end by a result on representation of compact group which will be useful in the sequel see. [6] Let $\widehat{K}$be the set of all equivalence (classes) finitedimensional irreducible representations of the compact group K. For $\delta \in \widehat{K}$, let $C(K/K_\Xi)(\delta)$ be the linear span of all K-finite functions on $K/K_\Xi$of type δ. Then, the algebraic sum $\bigoplus_{\delta \in \widehat{K}} C(K/K_\Xi)(\delta)$is dense in $C(K/K_\Xi)$under the topology of uniform convergence. Therefore $\bigoplus_{\delta \in \widehat{K}} C(K/K_\Xi)(\delta)$is dense in $L^p(K/K_\Xi)$.

Before giving the proof of our theorem of Fatou-type stated, we first show that the integral defining the c-function $\sum_{j=1}^{n} c(\lambda_j)$is absolutely convergent if $\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$.

For this, we recall a result on the partial Harish–Chandra c^Ξ-function (see [10]). Let $\mu \in a_c^\star$.Then, the integral representation of $c^\Xi$is given by

$$c^\Xi(\mu) = \int_{N_\Xi^-} e^{-(\mu+\rho)(H(n))} dn\,,$$

where $H(n) \in a$with respect to the Iwasawa decomposition $G = KAN$ of $G, x = \kappa(x) e^{H(x)} n(x)$for $x \in G$. The above integral converges absolutely (see [10]) if

$$\Re(\langle \mu, \alpha \rangle) > 0, \ \ \forall \alpha \in \Sigma^+ \backslash \langle \Xi \rangle.$$

Lemma (3): [13] *Let $\lambda_1, \lambda_2, \ldots \lambda_j \in \mathbb{C}$ such that $\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$. Then the integral*

$$\sum_{j=1}^{n} c(\lambda_j) = \int_{N_\Xi^-} \prod_{j=1}^{n} e^{-(\lambda_j \rho_0 + \rho_\Xi) H_\Xi(n)}\, dn\,.$$

converges absolutely.

Proof. Let $\lambda_1, \lambda_2, \ldots \lambda_j \in \mathbb{C}$ such that $\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$. Define the following $\mathbb{C}$-linear form $\sum_{j=1}^{n} \mu_{\lambda_j}$on a_c:

$$\sum_{j=1}^{n} \mu_{\lambda_j}(H) = \sum_{j=1}^{n} (\lambda_j \rho_0 - \rho_\Xi)(H_\Xi) + \rho(H),$$

where $H_\Xi$is the a_Ξ-component of $H$with respect to the orthogonal decomposition $a = a_\Xi \oplus a(\Xi)$. Notice that the condition

$\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$ is equivalent to $\sum_{j=1}^{n} \left(\langle \mu_{\lambda_j}, \alpha \rangle\right) > 0$ for all $\alpha \in \Sigma^+$.

Next let $w \in \{w \in W; wH = H, \forall H \in \mathfrak{a}_{\Xi}\}$ such that

(i) $w(\Sigma^+ \cap \langle \Xi \rangle) = -\Sigma^+ \cap \langle \Xi \rangle$ and

(ii) $w(\Sigma^+ \setminus \langle \Xi \rangle) = \Sigma^+ \setminus \langle \Xi \rangle$.

It is easy to see that $\sum_{j=1}^{n} w\mu_{\lambda_j} + \rho = \sum_{j=1}^{n} \lambda_j \rho_0 + \rho_{\Xi}$. Since $\sum_{j=1}^{n} \langle w\mu_{\lambda_j}, \alpha \rangle = \sum_{j=1}^{n} \langle \mu_{\lambda_j}, w^{-1}\alpha \rangle$ we get $\sum_{j=1}^{n} \Re\left(\langle w\mu_{\lambda_j}, \alpha \rangle\right) > 0, \forall \alpha \in \Sigma^+ \setminus \langle \Xi \rangle$ by (ii) and $\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$.

Hence the integral

$$\sum_{j=1}^{n} c^{\Xi}\left(w\mu_{\lambda_j}\right) = \int_{N_{\Xi}^-} \prod_{j=1}^{n} e^{-\left(w\mu_{\lambda_j}+\rho\right)(H(n))}\, dn,$$

converges absolutely and since the above integral is nothing but $\sum_{j=1}^{n} c(\lambda_j)$, the result follows.

For the proof of Theorem (4), we will need the following cocycle relations for the generalized Iwasawa function $H_{\Xi}(x)$:

$$H_{\Xi}(x\kappa(y)) = H_{\Xi}(xy) - H_{\Xi}(y) \qquad (4)$$

for all $x, y \in G$, and

$$H_{\Xi}(na^{-1}) = H_{\Xi}(n) - H_{\Xi}(a) \qquad (5)$$

for $n \in N_{\Xi}^-$ and $a \in A_{\Xi}$.

Theorem(4): *Let $\lambda_1, \lambda_2, \ldots \lambda_j \in \mathbb{C}$ such that $\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$. Then we have*:

$$f(k) = \sum_{j=1}^{n} c(\lambda_j)^{-1} \lim_{t\to+\infty} \prod_{j=1}^{n} e^{r(\eta-\lambda_j)t} P_{\lambda_j} f(ka_t),$$

(i) *uniformly for $f \in C(K/K_{\Xi})$;*

(ii) *in $L^p(K/K_{\Xi})$, if $f \in L^p(K/K_{\Xi}), 1 < p < +\infty$.*

Proof. (i) Let f in $C(K/K_{\Xi})$. Write $\sum_{j=1}^{n} P_{\lambda_j} f(ka_t)$ as

$$\sum_{j=1}^{n} P_{\lambda_j} f(ka_t) = \int_K \prod_{j=1}^{n} e^{-(\lambda_j+\eta)\rho_0(H_{\Xi}(a_{-t}h))} f(kh)dh.$$

Since the integrand

$$h \to \prod_{j=1}^{n} e^{-(\lambda_j+\eta)\rho_0(H_{\Xi}(a_{-t}h))} f(kh)$$

is a K_{Ξ}-invariant function in K, we can use the integral formula (3) to transform the above integral into an integral over N_{Ξ}^-:

$$\sum_{j=1}^{n} P_{\lambda_j} f(ka_t) = \int_{N_{\Xi}^-} \prod_{j=1}^{n} e^{-(\lambda_j+\eta)\rho_0(H_{\Xi}(a_{-t}h))}\, e^{-2\eta\rho_0(H_{\Xi}(n))} f(k\kappa(n))dn.$$

Next use (4) to get

$$\sum_{j=1}^{n} P_{\lambda_j} f(ka_t) = \int_{N_{\Xi}^-} \prod_{j=1}^{n} e^{-(\lambda_j+\eta)\rho_0(H_{\Xi}(a_{-t}h))}\, e^{(\lambda_j-\eta)\rho_0(H_{\Xi}(n))f(k\kappa(n))} dn.$$

Now, using on one hand the change of the variables $n \to a_{-t}na_t$ and on the other hand the identity (5), the above integral becomes

$$\prod_{j=1}^{n} e^{(\lambda_n-\eta)rt} \int_{N_{\Xi}^-} \prod_{j=1}^{n} e^{-(\lambda_j+\eta)\rho_0 H_{\Xi}(n) + (\lambda_j-\eta)\rho_0 H_{\Xi}(a_t n a_{-t})} f(k\kappa(a_t n a_{-t}))dn.$$

Since $a_t n a_{-t} \to e$, as t goes to $+\infty$, we deduce that

$$\lim_{t\to+\infty} \prod_{j=1}^{n} e^{(\eta-\lambda_j)rt} \left(P_{\lambda_j} f\right)(ka_t) = \sum_{j=1}^{n} c(\lambda_j)\, f(k),$$

provided we justify the reversal order of the limit and integration.

For this, let

$$\psi_t(n) = \prod_{j=1}^{n} e^{-(\lambda_j+\eta)\rho_0 H_{\Xi}(n) + (\lambda_j-\eta)\rho_0 H_{\Xi}(a_t n a_{-t})} f(k\kappa(a_t n a_{-t})).$$

Putting $M = \sup_{k\in K} |f(k)|$, we have

$$|\psi_t(n)| \le \prod_{j=1}^{n} e^{-(\Re\lambda_j+\eta)\rho_0 H_{\Xi}(n) + (\Re\lambda_j-\eta)\rho_0 H_{\Xi}(a_t n a_{-t})} M.$$

To end the proof we will need the following result which is a particular case of, [10]

Lemma (5): *Let $t > 0$and let $n \in N_{\Xi}^{-}$. Then we have*

$$0 \le \rho_0\big(H_{\Xi}(a_t n a_{-t})\big) \le \rho_0\big(H_{\Xi}(n)\big).$$

Using the above lemma, it is easy to see that $\psi_t$is dominated by the function

$$\begin{cases} \prod_{j=1}^{n} e^{-(\Re(\lambda_j)+\eta)\rho_0 H_{\Xi}}(n), & \text{if} -1 < \Re(\lambda_j) - \eta \le 0, \\ e^{-2\eta\rho_0 H_{\Xi}(n)}, & \text{if}\, \Re(\lambda_j) - \eta > 0 \end{cases}$$

which is integrable and the result follows.

For the proof of the L^p-counterpart of Theorem (4), we will need the following result.

Lemma (6): *Let $\lambda_1, \lambda_2, \dots \lambda_j \in \mathbb{C}$* [13] *such that $\sum_{j=1}^{n} \Re(\lambda_j) > \eta - 1$. Then, there exists a positive constant $\sum_{j=1}^{n} \gamma(\lambda_j)$ such that for $p > 1$ and $f \in L^p(K/K_{\Xi})$, we have*:

$$\left[\int_K \sum_{j=1}^{n} \left|P_{\lambda_j} f(ka_t)\right|^p dk\right]^{1/p} \le \prod_{j=1}^{n} \gamma(\lambda_j) e^{(\Re(\lambda_j)-\eta)rt} \|f\|_p. \qquad (6)$$

Proof. For fixed $t > 0$, define the function $\sum_{j=1}^{n} P_{\lambda_j}^t$on K by

$$\sum_{j=1}^{n} P_{\lambda\, j}^{t}(k) = \prod_{j=1}^{n} e^{-(\lambda_j+\eta)\rho_0 H_{\Xi}(a_{-t}k^{-1})}.$$

Then the Poisson integral $\sum_{j=1}^{n} P_{\lambda_j} f$can be written as a convolution over the compact group K:

$$\sum_{j=1}^{n} P_{\lambda_j} f(ka_t) = \sum_{j=1}^{n} f \star P_{\lambda\, j}^{t}(k).$$

Hence using the Hausdorff–Young inequality, we obtain

$$\left[\int_K \sum_{j=1}^{n} \left|P_{\lambda_j} f(ka_t)\right|^p dk\right]^{1/p} \le \sum_{j=1}^{n} \|f\|_p \left\|P_{\lambda\, j}^{t}\right\|_1.$$

Next, since

$$\sum_{j=1}^{n} \left\|P_{\lambda\, j}^{t}\right\|_1 = \int_K \prod_{j=1}^{n} e^{-(\Re(\lambda_j)+\eta)\rho_0 H_{\Xi}(a_{-t}k)}\, dk,$$

(i.e. $\sum_{j=1}^{n} \left\|P_{\lambda\, j}^{t}\right\|_1 = \sum_{j=1}^{n} P_{\Re(\lambda_j)} 1(a_t)$), we deduce from the part one of Theorem (4) that there exists a positiveconstant $\sum_{j=1}^{n} \gamma(\lambda_j)$ such that

$$\sum_{j=1}^{n} \left\|P_{\lambda\, j}^{t}\right\|_1 \le \prod_{j=1}^{n} \gamma(\lambda_j)\, e^{r(\Re(\lambda_j)-\eta)t},$$

which implies that

$$\sup_{t>0} \prod_{j=1}^{n} e^{r(\eta-\Re(\lambda_j))t} \left[\int_K \sum_{j=1}^{n} \left|P_{\lambda_j} f(ka_t)\right|^p dk\right]^{1/p} \le \sum_{j=1}^{n} \gamma(\lambda_j)\, \|f\|_p,$$

and the proof of Lemma (6) is finished.

Now, we give the proof of (ii) of Theorem (4). Let $f \in L^p(K/K_{\Xi})$. Then, for any $\epsilon > 0$, there exists $\phi \in \bigoplus_{\delta \in \widehat{K}} C(K/K_{\Xi})(\delta)$such that $\|f - \phi\|_p \le \epsilon$. We have

$$\prod_{j=1}^{n} \left\|c(\lambda_j)^{-1} e^{r(\eta-\lambda_j)t} P_{\lambda\, j}^{t}(f) - f\right\|_p \le \prod_{j=1}^{n} \left\|c(\lambda_j)^{-1} e^{r(\eta-\lambda_j)t} P_{\lambda\, j}^{t}(f - \phi)\right\|_p$$

$$+ \prod_{j=1}^{n} \left\|c(\lambda_j)^{-1} e^{(\eta-\lambda_j)rt} P_{\lambda\, j}^{t}\phi - \phi\right\|_p$$

$+\|\phi - f\|_p, \quad (7)$

where $\sum_{j=1}^{n} P_{\lambda\, j}^{t} f(k) = \sum_{j=1}^{n} P_{\lambda_j} f(ka_t)$.

The first term in the right-hand side of (7) is less then

$$\sum_{j=1}^{n} \gamma(\lambda_j) |c(\lambda_j)|^{-1} \|\phi - f\|_p,$$

by **Lemma (6).**

Since $\phi$is continuous on $K/K_{\Xi}$the (i) part of Theorem (4) shows that

$$\lim_{t\to+\infty}\prod_{j=1}^{n}\left\|c(\lambda_j)^{-1}e^{r(\eta-\lambda_j)t}P_{\lambda\,j}^{t}\phi-\phi\right\|_p=0.$$

Therefore,

$$\lim_{t\to+\infty}\prod_{j=1}^{n}\left\|c(\lambda_j)^{-1}e^{r(\eta-\lambda_j)t}P_{\lambda_j}f-f\right\|_p\leq\sum_{j=1}^{n}\epsilon(\gamma(\lambda_j)+1),$$

which implies (ii) and the proof of Theorem (4) is completed.

As a consequence of the L^p-Fatou-type theorem we get the following estimates on the Poisson transform on $L^p(K/K_{\Xi})$.

Corollary (7): *Let $\lambda_1,\lambda_2,\ldots\lambda_j$*[13] *be a complex number such that $\sum_{j=1}\Re(\lambda_j)>\eta-1$. There exists a positive constant $\sum_{j=1}^{n}\gamma(\lambda_j)$ such that for $1<p<+\infty$and $f\in L^p(K/K_{\Xi})$, we have*

$$\sum_{j=1}^{n}|c(\lambda_j)|\,\|f\|_p\leq\sum_{j=1}^{n}\left\|P_{\lambda_j}f\right\|_{\lambda_j,p}\leq\sum_{j=1}^{n}\gamma(\lambda_j)\,\|f\|_p.$$

Proof. We have only to prove the left-hand side of the above estimates. Let $f\in L^p(K/K_{\Xi})$. By (ii) of the previous theorem we know that

$$f(k)=\lim_{t\to+\infty}\prod_{j=1}^{n}c(\lambda_j)^{-1}\,e^{r(\eta-\lambda_j)t}P_{\lambda_j}f(ka_t),$$

in $L^p(K/K_{\Xi})$. Hence, there exists a sequence $(t_j)$with $t_j\to+\infty$ as $j\to+\infty$ such that

$$f(k)=\lim_{j\to\infty}\prod_{j=1}^{n}c(\lambda_j)^{-1}e^{r(\eta-\lambda_j)t_j}P_{\lambda_j}f\left(ka_{t_j}\right),$$

almost every where in K. By the classical Fatou lemma, we have

$$\|f\|_p\leq\prod_{j=1}^{n}|c(\lambda_j)|^{-1}e^{r(\eta-\Re(\lambda_j))t_j}\sup_j\left[\int_K\sum_{j=1}^{n}\left|P_{\lambda_j}f\left(ka_{t_j}\right)\right|^p dk\right]^{1/p},$$

which gives $\|f\|_p\leq\sum_{j=1}^{n}|c(\lambda_j)|^{-1}\left\|P_{\lambda_j}f\right\|_{\lambda_j,p}$ and the proof of Corollary (3.3.7) is finished.

Recall that the group Kacts on $L^2(K/K_{\Xi})$by $\pi(h)f(k)=f(h^{-1}k)$and under this action, the space $L^2(K/K_{\Xi})$has the following Peter–Weyl decomposition $L^2(K/K_{\Xi})=\bigoplus_{\delta\in\widehat{K_0}}V_\delta$,where $\widehat{K_0}$ denotes the set of all class one (with respect to K_{Ξ}) equivalence-classes-irreducible representations of $K$and $V_\delta$is the finite linear span of $\{\phi_\delta\circ k;k\in K\},\phi_\delta$being the zonal spherical function.

Proposition (8): *Let $\lambda_1,\lambda_2,\ldots\lambda_j\in\mathbb{C}$ and let $f\in V_\delta$. Then*

$$\sum_{j=1}^{n}P_{\lambda_j}f(ka_t)=\sum_{j=1}^{n}\Phi_{\lambda_j,\delta}(a_t)f(k),$$

where $\sum_{j=1}^{n}\Phi_{\lambda_j,\delta}(a_t)=\sum_{j=1}^{n}P_{\lambda_j}\phi_\delta(a_t)$.

Proof. For each $t\in\mathbb{R}$, define the operator $\sum_{j=1}^{n}P_{\lambda\,j}^{t}$on $L^2(K/K_{\Xi})$ by $\sum_{j=1}^{n}P_{\lambda\,j}^{t}f(k)=\sum_{j=1}^{n}P_{\lambda_j}f\,(ka_t)$. Since M_{Ξ}centralizes $A_{\Xi},\sum_{j=1}^{n}P_{\lambda\,j}^{t}$defines a bounded operator in $L^2(K/K_{\Xi})$. Also, we can see easily that $\sum_{j=1}^{n}P_{\lambda_j}^{t}$commutes with π. Hence, by Schur lemma

$$\sum_{j=1}^{n}P_{\lambda\,j}^{t}=\sum_{j=1}^{n}\Phi_{\lambda_j,\delta}(a_t)\,I$$

on each V_δ, with $\sum_{j=1}^{n}\Phi_{\lambda_j,\delta}(a_t)=\sum_{j=1}^{n}P_{\lambda_j}^{t}\phi_\delta\,(e)$.

From Theorem (4) we deduce the following corollary given the asymptotic behaviour of the generalized spherical function $\sum_{j=1}^{n}\Phi_{\lambda_j,\delta}$.

Corollary (9): *Let $\lambda_1,\lambda_2,\ldots\lambda_j\in\mathbb{C}$ such that $\sum_{j=1}^{n}\Re(\lambda_j)>\eta-1$. Then*

$$\lim_{t\to+\infty}e^{r(\eta-\lambda)t}\Phi_{\lambda,\delta}(a_t)=c(\lambda)$$

for each $\delta\in\widehat{K_0}$.

Theorem (10): *Let λ be a complex number such that $\sum_{j=1}^{n}\Re(\lambda_j)>\eta-1$. Then we have*:

(i) *A $\mathbb{C}$-valued function F on X satisfying the Hua system* (2) *is the Poisson transform by P_λ of some $f\in L^2(K/K_\Xi)$ if and only if it satisfies $\sum_{j=1}^{n}\|F\|_{\lambda_j,2}<+\infty$.*

Moreover, there exists a positive constant $\sum_{j=1}^{n}\gamma(\lambda_j)$ such that for $f\in L^2(K/K_\Xi)$ the following estimates hold:

$$\sum_{j=1}^{n}|c(\lambda_j)|\|f\|_2\le\sum_{j=1}^{n}\left\|P_{\lambda_j}f\right\|_{\lambda_j,2}\le\sum_{j=1}^{n}\gamma(\lambda_j)\|f\|_2. \quad (8)$$

(ii) *Let $F\in\sum_{j=1}^{n}E^{*}_{\lambda_j,2}(X)$. Then its L^2-boundary value f is given by the following inversion formula*:

$$f(k)=\sum_{j=1}^{n}\left|c(\lambda_j)\right|^{-2}\lim_{t\to+\infty}\prod_{j=1}^{n}e^{2r(\eta-\Re(\lambda_j))t}\int_K\overline{e^{-(\lambda_j\rho_0+\rho_\Xi)H_\Xi(a_{-t}k^{-1}h)}}F(ha)\,dh,$$

in $L^2(K/K_\Xi)$.

Proof.

(i) The necessary condition follows from Lemma (6), for $p=2$.

To prove the sufficiency condition, let $F\in\sum_{j=1}^{n}E^{\star}_{\lambda_j,2}(X)$. Since $\sum_{j=1}^{n}\Re(\lambda_j)>\eta-1$, we have $F=\sum_{j=1}^{n}P_{\lambda_j}f$ for some $f\in B(K/K_\Xi)$, by Shimeno result.

Let $f=\sum_{\delta\in\widehat{K_0}}f_\delta$ be its K-type series. Then, using Proposition (8), F can be written as

$$F(ka_t)=\sum_{\delta\in\widehat{K_0}}\sum_{j=1}^{n}\Phi_{\lambda_j,\delta}(a_t)f_\delta(k)$$

in $C^\infty(K\times[0,+\infty))$.

Next, since $\sum_{j=1}^{n}\|F\|_{\lambda_j,2}<+\infty$, we have

$$\prod_{j=1}^{n}e^{2r(\eta-\Re(\lambda_j))t}\sum_{\delta\in\widehat{K_0}}\sum_{j=1}^{n}\left|\Phi_{\lambda_j,\delta}(a_t)\right|^2\|f_\delta\|_2^2<+\infty$$

for every $t>0$.

Let Ω be a finite subset of $\widehat{K_0}$. Then, using the asymptotic behaviour of $\sum_{j=1}^{n}\Phi_{\lambda_j,\delta}$ given by Corollary (9), we see that

$$\sum_{j=1}^{n}\left|c(\lambda_j)\right|^2\sum_{\delta\in\Omega}\|f_\delta\|_2^2\le\sum_{j=1}^{n}\|F\|^2_{\lambda_j,2}.$$

Since Ω is arbitrary, it follows that $f=\sum_{\delta\in\widehat{K_0}}f_\delta\in L^2(K/K_\Xi)$ and that $\sum_{j=1}^{n}\left|c(\lambda_j)\right|\|f\|_2\le\sum_{j=1}^{n}\left\|P_{\lambda_j}f\right\|_{\star,2}$. This finishes the proof of the first part of Theorem (10).

(ii) Now, we turn to the proof of the L^2-inversion formula.

Let $F\in\sum_{j=1}^{n}E^{\star}_{\lambda_j,2}(X)$. By the first part of Theorem (10), we know that there exists a unique $f\in L^2(K/K_\Xi)$ such that $F=\sum_{j=1}^{n}P_{\lambda_j}f$. Hence, expanding f into its K-type series, $f=\sum_{\delta\in\widehat{K_0}}f_\delta$, Proposition (8) shows that

$$F(ka_t)=\sum_{\delta\in\widehat{K_0}}\sum_{j=1}^{n}\Phi_{\lambda_j,\delta}(a_t)f_\delta(k)$$

in $C^\infty(K\times[0,+\infty[)$.

Next, for each $t>0$ we define a $\mathbb{C}$-valued function g_t on K by

$$g_t(k)=|c(\lambda)|^{-2}e^{2r(\eta-\Re(\lambda))t}\int_K\overline{e^{-(\lambda+\eta)\rho_0H_\Xi(a_{-t}k^{-1}h)}}F(ha)dh.$$

Then, replacing F by its series expansion and using again Proposition (8), we see that g_t can be rewritten as

$$g_t(k)=\prod_{j=1}^{n}\left|c(\lambda_j)\right|^{-2}e^{2(\eta-\Re(\lambda_j))rt}\sum_{\delta\in\widehat{K_0}}\sum_{j=1}^{n}\left|\Phi_{\lambda_j,\delta}(a_t)\right|^2f_\delta(k)$$

in $C^\infty(K)$.

Now from

$$\|g_t-f\|_2^2=\sum_{\delta\in\widehat{K_0}}\prod_{j=1}^{n}\left|\left|c(\lambda_j)\right|^{-2}e^{2(\eta-\Re\lambda_j)rt}\left|\Phi_{\lambda_j,\delta}(a_t)\right|^2-1\right|^2\|f_\delta\|_2^2,$$

and $\lim_{t\to+\infty}\prod_{j=1}^{n}e^{(\eta-\lambda_j)rt}\Phi_{\lambda_j,\delta}=\sum_{j=1}^{n}c(\lambda_j)$, we deduce that $\lim_{t\to+\infty}\|g_t-f\|_2=0$, and the proof of Theorem (10) is completed.

Theorem (11): *Let $\lambda_1,\lambda_2,\dots\lambda_j\in\mathbb{C}$ such that $\sum_{j=1}^{n}\Re(\lambda_j)>\eta-1$, and let $p, 1<p<+\infty$. Then we have a function $F\in\sum_{j=1}^{n}E_{\lambda_j}(X)$ is the Poisson transform by*

$\sum_{j=1}^{n} P_{\lambda_j}$ of some $f \in L^p(K/K_\Xi)$ if and only if $F \in \sum_{j=1}^{n} E^*_{\lambda_j,p}(X)$.

Moreover, there exists a positive constant $\sum_{j=1}^{n} \gamma(\lambda_j)$ such that for $f \in L^p(K/K_\Xi)$ the following estimates hold:

$$\sum_{j=1}^{n} |c(\lambda_j)| \|f\|_p \le \sum_{j=1}^{n} \left\|P_{\lambda_j} f\right\|_{\lambda_j,p} \le \sum_{j=1}^{n} \gamma(\lambda_j) \|f\|_p. \qquad (9)$$

Proof.

The "if" part follows from Lemma (6).

The proof of the converse will be divided into two parts.

(i) The case $p \ge 2$. Firstly, observe that in this case $\sum_{j=1}^{n} E^\star_{\lambda_j,p}(X) \subset \sum_{j=1}^{n} E^\star_{\lambda_j,2}(X)$. Hence, for a given $F \in \sum_{j=1}^{n} E^\star_{\lambda_j,p}(X)$, we know by Theorem (10) that there exists $f \in L^2(K/K_\Xi)$ such that $F = \sum_{j=1}^{n} P_{\lambda_j} f$ and that the function f can be recovered from F via the L^2-type inversion formula $f(k) = \lim_{t\to+\infty} g_t(k)$ in $L^2(K)$, where

$$g_t(k) = \prod_{j=1}^{n} |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \int_K \prod_{j=1}^{n} \overline{e^{-(\lambda_j+\eta)\rho_0 H_\Xi(a_{-t}k^{-1}h)}} F(ha_t)dh.$$

Let ϕ be a continuous function in K/K_Ξ. Then we have

$$\lim_{t\to+\infty} \int_K g_t(k)\overline{\phi(k)}dk = \int_K f(k)\overline{\phi(k)}dk.$$

But

$$\int_K g_t(k)\overline{\phi(k)}dk = \prod_{j=1}^{n} |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \int_K \left[\int_K \prod_{j=1}^{n} \overline{e^{-(\lambda_j+\eta)\rho_0 H_\Xi(a_{-t}h^{-1}k)}} F(ha_t)dh\right] \overline{\phi(k)}dk.$$

Observing that

$$H_\Xi(a_t k) = H_\Xi(a_t k^{-1}),$$

for every $k \in K$ and using Fubini theorem, we can rewrite the right-hand side of the above equality as

$$\prod_{j=1}^{n} |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \int_K \prod_{j=1}^{n} \overline{P_{\lambda_j}\phi(ha_t)} F(ha_t)dh,$$

which is—by the Hölder inequality—majorized by

$$\prod_{j=1}^{n} |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \left[\int_K \sum_{j=1}^{n} \left|P_{\lambda_j}\phi(ha_t)\right|^q dh\right]^{1/q} \left[\int_K |F(ha_t)|^p dh\right]^{1/p},$$

where q is such that $1/q + 1/p = 1$.

Since $F \in \sum_{j=1}^{n} E^\star_{\lambda_j,p}(X)$, we obtain

$$\left|\int_K g_t(k)\overline{\phi(k)}dk\right| \le \prod_{j=1}^{n} |c(\lambda_j)|^{-2} e^{2r(\eta - \Re(\lambda_j))t} \sum_{j=1}^{n} \left[\int_K \left|P_{\lambda_j}\phi(ha_t)\right|^q dh\right]^{1/q} \|F\|_{\lambda_j,p}.$$

By Theorem (10) we know that

$$\phi(k) = \sum_{j=1}^{n} c(\lambda_j)^{-1} \lim_{t\to+\infty} \prod_{j=1}^{n} e^{r(\eta-\lambda_j)t} P_{\lambda_j}\phi(ka_t)$$

in $L^q(K/K_\Xi)$. Hence

$$\left|\int_K f(k)\overline{\phi(k)}dk\right| \le \sum_{j=1}^{n} |c(\lambda_j)|^{-1} \|\phi\|_q \|F\|_{\lambda_j,p}.$$

Finally, taking the supremum over all continuous ϕ with $\|\phi\|_q = 1$ in the above inequality we deduce that $f \in L^p(K/K_\Xi)$ and that $\sum_{j=1}^{n} |c(\lambda_j)| \|f\|_p \le \sum_{j=1}^{n} \|F\|_{\lambda_j,p}$, which is the desired result.

(ii) Part 2. The case $1 < p \le 2$. Let χ_n be an approximation of the identity in $C(K)$. That is,

$$\chi_n \ge 0, \quad \int_K \chi_n(k)\, dk = 1 \quad \text{and} \lim_{n\to+\infty} \int_{K/V} \chi_n(k)dk = 0$$

for every neighborhood V of e in K.

Put

$$F_n(g) = \int_K \chi_n(k) F(k^{-1}g)dk.$$

Then, $\lim_{n\to+\infty} F_n = F$ point wise in G. Since the eigenspace $\sum_{j=1}^{n} E_{\lambda_j}(X)$ is G-invariant, F_n lies also in $\sum_{j=1}^{n} E_{\lambda_j}(X)$. For each $t > 0$ define a function F_n^t in K by $F_n^t(k) = F_n(ka_t)$. Then $F_n^t = \chi_n \star F^t$. Moreover, we have

$$\|F_n^t\|_2 \le \|\chi_n\|_2\|F^t\|_1 \le \|\chi_n\|_2\|F^t\|_p.$$

From the above inequalities we see that for each $n$the defined functions F_n lies in the space $\sum_{j=1}^{n} E_{\lambda_j,2}^{\star}(X)$. Hence, there exists $f_n \in L^2(K/K_\Xi)$ such that $F_n = \sum_{j=1}^{n} P_{\lambda_j} f_n$, by Theorem (10).

Let q be a positive number such that $1/p + 1/q = 1$ and let T_n be the linear form defined in $L^q(K/K_\Xi)$ by

$$T_n(\phi) = \int_K f_n(k)\phi(k)dk\,.$$

Since $p \le 2$, we have $f_n \in L^p(K/K_\Xi)$. Thus, the linear form T_n is continuous and

$$|T_n(\phi)| \le \|f_n\|_p\|\phi\|_q.$$

By Corollary (7), we have $\|f_n\|_p \le \sum_{j=1}^{n}|c(\lambda_j)|^{-1}\left\|P_{\lambda_j} f_n\right\|_{\lambda_j,p}$.

Hence,

$$|T_n(\phi)| \le \sum_{j=1}^{n}|c(\lambda_j)|^{-1}\|F_n\|_{\lambda_j,p}\|\phi\|_q\,.$$

Now from

$$\|F_n^t\|_p \le \|\chi\|_1\|F^t\|_p = \|F^t\|_p,$$

we deduce that $\sum_{j=1}^{n}\|F_n\|_{\lambda_j,p} \le \sum_{j=1}^{n}\|F\|_{\lambda_j,p}$ and this implies clearly that

$$|T_n\phi)| \le \sum_{j=1}^{n}|c(\lambda_j)|^{-1}\|F\|_{\lambda_j,p}\|\phi\|_q\,.$$

Thus, the linear forms T_n are uniformly bounded operators in $L^q(K/K_\Xi)$, with

$$\sup_n\|T_n\| \le \sum_{j=1}^{n}|c(\lambda_j)|^{-1}\|F\|_{\lambda_j,p}\,,$$

where $\|\cdot\|$stands for the operator norm.

Next, use the Banach–Alaoglu–Bourbaki theorem to conclude that there exists a subsequence of bounded operators (T_{n_j})which converges as $n_j \to +\infty$to a bounded operator T on $L^q(K/K_\Xi)$, under the $\star$-weak topology, with $\|T\| \le \sum_{j=1}^{n}|c(\lambda_j)|^{-1}\|F\|_{\lambda_j,p}$. By the Riesz representation theorem, we know that there exists a unique function $f \in L^p(K/K_\Xi)$such that

$$T(\phi) = \int_K f(k)\phi(k)dk\,.$$

Now, let

$$\phi_g(k) = \prod_{j=1}^{n} e^{-(\lambda_j+\eta)\rho_0 H_\Xi(g^{-1}k)}.$$

Then, $T_n(\phi_g) = F_n(g)$. Since, on the one hand,

$$\lim_{n\to+\infty} F_n(g) = F(g)$$

and, on the other hand,

$$\lim_{j\to+\infty} T_{n_j}(\phi_g) = T(\phi_g),$$

we get $F(g) = \sum_{j=1}^{n} P_{\lambda_j} f(g)$. The estimate $\|f\|_p \le \sum_{j=1}^{n}|c(\lambda_j)|^{-1}\|F\|_{\lambda_j,p}$ follows obviously from the bound of $T$and the proof of Theorem (11) is finished.

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How to cite this article: Mohammed Rabih NA, Suhail DSM, Osman Abdalla Adam Osman. Solutions of the hua system on hermitian symmetric spaces of tube type. International Journal of Research and Review. 2017; 4(4):29-39.

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