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Research Paper

Isomorphic Classification Sequences of $C(U_n, X)$ Spaces

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ABSTRACT

In this paper we study the isomorphic classifications sequences of $C(U_n, X)$spaces, the Banach spaces of all continuous X-valued functions defined on infinite compact sequences of metric spacesU_n, equipped with the supremum norm. We first introduce the concept of $\alpha + \varepsilon$ -quotient of Banach spaces X. Thus, we prove that if X has some $\alpha + \varepsilon$ -quotient which is uniformly convex, then for all $U_{n+1}$and U_{n+2} the following statements are equivalent:
(a) $C(U_{n+1}, X)$is isomorphic to $C(U_{n+2}, X)$.
(b) $C(U_{n+1})$is isomorphic to $C(U_{n+2})$.
This allows us to classify, up to an isomorphism, some $C(U_n, Y \oplus l_p(\Gamma))$spaces, $1 < p \leq \infty$, and certain $C(S_n)$ spaces involving large compact Hausdorff sequence spaces S_n.

Keywords: Bessaga-Pełczyński and Milutin's theorems on separable $C(K)$spaces Isomorphic classifications of $C(K, X)$ spaces ω_1-quotient of Banach spaces

1. INTRODUCTION

We refer the reader to [1,7,18,19] for details on standard notation and terminology we use in the paper. For a compact Hausdorff topological sequence spaces U_n let $C(U_n, X)$denote the Banach space of all continuous X-valued functions defined on U_n , equipped with the supremum norm. This space will be denoted by $C(U_n)$in the case where $X = R$. As usual, in the case where U_n is the interval of ordinals $[0, \alpha]$endowed with the order topology, these spaces will be denoted respectively by $C(\alpha, X)$and $C(\alpha)$. When α is the first infinite ordinal, these spaces will be also denoted by $c_0(X)$and $c_0$respectively. If U_n and S_n are compact Hausdorff sequences spaces, we denote by $U_n \oplus S_n$ and $U_n \times S_n$ respectively the topological sum and the topological product of $U_n$and S_n. For a fixed cardinal number $\varepsilon > 0$, $\mathbf{2}^{1+\varepsilon}$denotes the Cantor cube, that is, the product of mfamily of copies of the two-point space $\mathbf{2}$, provided with the product topology. If X and Y are Banach spaces, then $X \sim Y$ means that X is isomorphic to Y and $X \twoheadrightarrow Y$ means that $Y$is isomorphic to a quotient of X. Finally, the symbol $X \oplus Y$ denotes the Cartesian product of $X$and Y.

The central result on the isomorphic classification of separable sequences of $C(U_n)$spaces, that is,$U_n$are metric spaces, is Milutin's Theorem, [13] see. [15-17] This result states that if $U_n$is an uncountable compact sequences of metric space, then

$$C(U_n) \sim C(\mathbf{2}^{\aleph_0}). \qquad (1.1)$$

In the case where $U_n$is a countable compact sequences of metric space, a classical

Mazurkiewicz and Sierpiński's Theorem [12] asserts that U_n is homeomorphic to some interval of ordinals $[0,\alpha]$ for some ordinal ε, where $\alpha+\varepsilon$ is the first uncountable ordinal. The isomorphic classification of the $C(\alpha)$ spaces was done by Bessaga and Pełczyński [2] in the following way. Let ξ and η be two ordinals such that $\omega \le \xi \le \eta < \alpha+\varepsilon$. Then

$$C(\xi) \sim C(\eta) \Leftrightarrow \eta < \xi^{\omega}. \qquad (1.2)$$

In this section we are mainly interested in getting the isomorphic classification of certain spaces involving the spaces (1.1) and (1.2). The starting point of our research is the fact that recently in [10] it was provided an extension of (1.2) to the vector-valued case. Namely, recall that a subspace H of a Banach space X is a maximal factor of X whenever X is the direct sum of H and some subspace Y of X such that every finite sum Y^n of Y contains no copy of H. Then, the main result of [10] is as follows.

Theorem 1.1. Let X be a Banach space containing some uniformly convex maximal factor and ordinals $\omega \le \xi \le \eta < \alpha+\varepsilon$. Then

$$C(\xi, X) \sim C(\eta, X) \Leftrightarrow \eta < \xi^{\omega}.$$

Of course Theorem1.1 can be applied to obtain the isomorphic classifications of so many $C(\alpha, X)$ spaces, where $\omega \le \alpha < \alpha+\varepsilon$. In particular, since $C(\mathbf{2}^{1+\varepsilon})$ contains no copy of the classical uniformly convex Banach spaces $l_{(q-\varepsilon)}(\Gamma)$, $1 < (q-\varepsilon) < \infty$, whenever Γ is an uncountable set [5,14] and moreover

$$C(\alpha, C(\mathbf{2}^{1+\varepsilon})) \sim C(\mathbf{2}^{1+\varepsilon}), \qquad (1.3)$$

for all $\omega \le \alpha < \alpha+\varepsilon$ *and* infinite cardinal $\varepsilon > 0$, it follows by Theorem1.1 that the isomorphic classification of the following spaces is the same as that of $C(\alpha)$ spaces, $\omega \le \alpha < \alpha+\varepsilon$, mentioned in (1.2)

$$C\big(\alpha, C(\mathbf{2}^{1+\varepsilon}) \oplus l_{(q-\varepsilon)}(\Gamma)\big) \sim C(\mathbf{2}^{1+\varepsilon}) \oplus C\big(\alpha, l_{(q-\varepsilon)}(\Gamma)\big), \qquad (1.4)$$

On the other hand, observe that when Γ is finite, the spaces (1.4) are isomorphic to $C(\mathbf{2}^{1+\varepsilon})$, for all $\omega \le \alpha < \alpha+\varepsilon$ *and* infinite cardinal $\varepsilon > 0$.

Then, it is natural to look for the complete isomorphic classification of the spaces (1.4)when $1 \le (q-\varepsilon) \le \infty$. The study of this question in the case where $(q-\varepsilon) \ne 1$ *led* us to obtain two more general isomorphic classifications of some $C(U_n, X)$ spaces for infinite compact sequences of metric spaces U_n. So, our contribution to answering the above question will be presented as a consequence of them. More precisely, in Section 3 we will prove:

Theorem 1.2. Let Y be a Banach space, $1 < (q-\varepsilon) < \infty$ and Γ be aninfinite set. Suppose that Y^* contains no copy of l_q, where $1/(q-\varepsilon) + 1/q = 1$. Then for all infinite compact sequences of metric spaces U_{n+1} and U_{n+2},

$$C(U_{n+1}, Y \oplus l_{(q-\varepsilon)}(\Gamma)) \sim C(U_{n+2}, Y \oplus l_{(q-\varepsilon)}(\Gamma)) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2}).$$

Therefore in the case where $1 < (q-\varepsilon) < 2$, since the dual of each $C(\mathbf{2}^{1+\varepsilon})$ space contains no copy of l_q, with $q > 2$, [1] the isomorphic classification of the spaces (1.4)with $1 < (q-\varepsilon) < 2$ is a corollary of Theorem1.2 regardless of whether the infinite set Γ is countable or uncountable. This furnishes a solution to [10] when $1 < (q-\varepsilon) < 2$.

Furthermore, recall that the density character of a topological space F (denoted by *densF*) is the smallest cardinality of a dense subset of F and denote by $|\Gamma|$ the cardinality of a set Γ. In Section 4 we will prove the following theorem.

Theorem 1.3. Let Y be a Banach space and Γ an infinite set. Suppose that dens $Y < 2|\Gamma|$. Then for all infinite compact sequences of metric spaces U_{n+1} and U_{n+2},

$$C(U_{n+1}, Y \oplus l_{\infty}(\Gamma)) \sim C(U_{n+2}, Y \oplus l_{\infty}(\Gamma)) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2}).$$

Thus, since $densC(\mathbf{2}^{1+\varepsilon}) = (1+\varepsilon)$, for every infinite cardinal m, [18] Theorem1.3 provides the isomorphic classification of the spaces (1.4)when $(q-\varepsilon) = \infty$ and $\aleph_0 \leq (1+\varepsilon) < 2^{|\Gamma|}$. In the case where $(1+\varepsilon) = |\Gamma| = \aleph_0$, Theorem(1.3) solves. [10]

In order to prove Theorems1.2 and 1.3, in the next section we state our main result (Theorem 2.4) which is a suitable extension of Theorem1.1.

2. The Isomorphic Classification of Certain Sequence of $C(U_n, X)$spaces

Concerning Theorem 1.1 our main technical improvement in this section is to replace the uniformly convex maximal factor of X by a similarly positioned subspace of Xwhich has a uniformly convex quotient. We start by introducing the following definition:

Definition 2.1. We say that a Banach space Z is an $\alpha+\varepsilon$-quotient of a Banach space Xif there exist subspaces A and B of Xsuch that

(a)$X = A \oplus B$,

(b)$B \twoheadrightarrow Z$,

(c)$C(\xi, A) \oplus B^n \nrightarrow c_0(Z)$, for every $\omega \leq \xi < \alpha+\varepsilon$ and $1 \leq n < \omega$.

Remark 2.2. The above definition was inspired by the proof of. [9] This result states that if $F$is the uniformly convex Banach space introduced by Figiel in [8] and $Z = F^*$, then for all ordinals $\omega \leq \xi \leq \eta < \alpha+\varepsilon$, $C(\xi, C(\mathbf{2}^{\aleph_0}) \oplus Z) \sim C(\eta, C(\mathbf{2}^{\aleph_0}) \oplus Z) \Leftrightarrow \eta < \xi^\omega$.

In order to prove this, it was shown that for all $1 \leq n < \omega$,

$$C(\mathbf{2}^{\aleph_0}) \oplus Z^n \nrightarrow c_0(Z). \qquad (2.1)$$

Thus, we can see Definition 2.2 as a refinement of this technical obstruction to maps onto c_0 sums. Indeed, according to (1.3) and (2.1) we deduce that the dual of the Figiel space $F$is an $\alpha+\varepsilon$-quotient of $C(\mathbf{2}^{\aleph_0}) \oplus F^*$.

Remark 2.3. Notice that $\alpha+\varepsilon$-quotients of a Banach space $X$are in fact quotients of X; while $l_1$is not an $\alpha+\varepsilon$-quotient of itself. Moreover, any Banach space Zcontaining no quotient isomorphic to $c_0$is an $\alpha+\varepsilon$-quotient of itself. Indeed, if the item (c) of Definition 2.2 does not hold with $A = 0$ and $B = Z$, then

$$Z^n \twoheadrightarrow c_0(Z) \twoheadrightarrow c_0,$$

for some $1 \leq n < \omega$. Therefore by [17] c_0 is isomorphic to a quotient of Z, which is an absurd. In particular, each uniformly convex space is an $\alpha+\varepsilon$-quotient of itself.

The aim of this section is to prove the following isomorphic classification.

Theorem 2.4. Let Xbe a Banach space having an $\alpha+\varepsilon$-quotient which is uniformly convex. Then for all infinite compact sequences of metric spaces $U_{n+1}$and U_{n+2},

$$C(U_{n+1}, X) \sim C(U_{n+2}, X) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2}).$$

Before proving this theorem, we shall state two propositions.

Proposition 2.5. Let A, B and Z be Banach spaces such that $Z$is uniformly convex and ordinals $\omega \leq \xi \leq \eta < \alpha+\varepsilon$. Suppose that

(a)$B \twoheadrightarrow Z$,

(b)$A \oplus B^n \nrightarrow c_0(Z)$, for every $1 \leq n < \omega$.

Then

$$A \oplus C(\xi, B) \twoheadrightarrow C(\eta, Z) \Longrightarrow \eta < \xi^\omega.$$

Proof. First we will show by transfinite induction that for any $0 \leq \alpha < \alpha+\varepsilon$ and $\varepsilon_1 > 0$

$$A \oplus C(\gamma, B) \nrightarrow C(\gamma+\varepsilon_1, Z). \qquad (2.2)$$

The hypothesis (b) covers the case $\alpha = 0$. Next suppose that $\varepsilon_2 = 1$, for some ordinal α, and for all $\varepsilon_1 > 0$ (2.2)holds. Assume that

$$A \oplus C(\gamma_1, B) \twoheadrightarrow C(\omega^{\omega^\beta}, Z) = C((\gamma+\varepsilon_1)^\omega, Z), \qquad (2.3)$$

for some $\gamma_1 < \omega^{\omega^\beta}$.

Now observe that if $\gamma_1 < \gamma+\varepsilon_1$ then $C(\gamma+\varepsilon_1, B) \twoheadrightarrow C(\gamma_1, B)$. Moreover, if $\gamma+\varepsilon_1 \leq \gamma_1$, then by (1.2)we have $C(\gamma+\varepsilon_1, B) \sim C(\gamma_1, B)$. Thus, by (2.3)

$$A \oplus C(\gamma+\varepsilon_1, B) \twoheadrightarrow C((\gamma+\varepsilon_1)^\omega, Z).$$

Therefore by [11] there exists an ordinal $\gamma_2 < \gamma+\varepsilon_1$ such that

$$A \oplus C(\gamma_2, B) \twoheadrightarrow C(\gamma+\varepsilon_1, Z),$$

but this contradicts (2.2).

Finally suppose that $\alpha+\varepsilon_2$is a limit ordinal and for all $0 < \varepsilon_2$ and $\varepsilon_1 > 0$ (2.2)holds.

Assume that

$$A \oplus C(\gamma_1, B) \twoheadrightarrow C(\gamma_1+\varepsilon_3, Z), \qquad (2.4)$$

for some $0 < \varepsilon_3$. Pick an ordinal α such that $\gamma_1 < \gamma + \varepsilon_1 < \gamma_1 + \varepsilon_3$. According to (2.4)
$A \oplus C(\gamma_1, B) \twoheadrightarrow C(\gamma + \varepsilon_1, Z)$,
contradicting (2.2).

Now we pass to prove the statement of the proposition. Assume then that

$$A \oplus C(\xi, B) \twoheadrightarrow C(\eta, Z), \qquad (2.5)$$

with $\omega \le \xi \le \eta < \alpha + \varepsilon$.

In view of (1.2) the spaces $C(\omega^{\omega^\gamma})$, for $0 \le \gamma < \alpha + \varepsilon$, are a complete set of representatives of the isomorphism classes of $C(\xi)$spaces for $0 \le \xi < \alpha + \varepsilon$. So, let αbe the ordinal such that
$C(\eta) \sim C(\gamma + \varepsilon_1)$.

Notice that $0 < \varepsilon_4$ and

$$C(\eta, Z) \sim C(\gamma + \varepsilon_1, Z). \qquad (2.6)$$

According to (2.5) and (2.6)

$$A \oplus C(\xi, B) \twoheadrightarrow C(\gamma + \varepsilon_1, Z). \qquad (2.7)$$

Hence by (2.2) and (2.7)we have $\gamma + \varepsilon_1 \le \xi$and therefore $\eta + \varepsilon_4 \le \xi^\omega$. Consequently $\eta < \xi^\omega$. □

Remark 2.6. Suppose that Z is isomorphic to a quotient of the Banach space B. It follows from the Bartle–Graves continuous selection for quotient maps [4] [p.52] that $C(\xi, Z)$is isomorphic to a quotient of $C(\xi, B)$for every ordinal ξ (See in [19]).

Proposition 2.7. Let X be a Banach space having an $\alpha + \varepsilon$-quotient which is uniformly convex. Then for all ordinals $\omega \le \xi \le \eta < \alpha + \varepsilon$,

$$C(\xi, X) \sim C(\eta, X) \Rightarrow \eta < \xi^\omega.$$

Proof. By hypothesis there exist a uniformly convex space $Z$and subspaces A *and* $B$of X satisfying (a), (b) and (c) of Definition 2.1. First of all observe that if we fix an ordinal $\omega \le \xi_0 < \alpha + \varepsilon$, since
$C(\xi_0, A) \oplus B^n \twoheadrightarrow c_0(Z)$,
for every $1 \le n < \omega$, it follows from Proposition2.5 applied to the spaces $C(\xi_0, A)$, Band Zthat for all ordinals $\omega \le \xi \le \eta < \alpha + \varepsilon$,

$$C(\xi_0, A) \oplus C(\xi, B) \twoheadrightarrow C(\eta, Z) \Longrightarrow \eta < \xi^\omega. \qquad (2.8)$$

Now, pick ordinals $\omega \le \xi \le \eta < \alpha + \varepsilon$ and suppose that

$$C(\xi, X) \sim C(\eta, X). \qquad (2.9)$$

Since $X = A \oplus B$ and $B \twoheadrightarrow Z$, by (2.9) we have
$C(\xi, A) \oplus C(\xi, B) \sim C(\eta, A) \oplus C(\eta, B) \twoheadrightarrow C(\eta, Z)$.

According to (2.8) with $\xi_0 = \xi$ we obtain $\eta < \xi^\omega$. □

Now we are ready to prove the main result of this paper.

Proof of Theorem 2.4.The condition is clearly sufficient. Let us show necessity. Suppose then that $C(U_{n+1}, X)$is isomorphic to $C(U_{n+2}, X)$, for some infinite compact sequences of metric spaces $U_{n+1}$and U_{n+2}. We distinguish two cases:

Case 1. U_{n+1} and $U_{n+2}$are countable. Let $\xi$and ηbe infinite countable ordinals such that $C(U_{n+1})$is isomorphic to $C(\xi)$and $C(U_{n+2})$is isomorphic to $C(\eta)$. Hence
$C(\xi, X) \sim C(\eta, X)$.

Without loss of generality we may assume that $\xi \le \eta$. So, by Proposition 2.7 and (1.2) we infer that $C(U_{n+1})$is isomorphic to $C(U_{n+2})$.

Case 2. U_{n+2} is uncountable. In this case, by (1.1) it suffices to show that $U_{n+1}$is also uncountable. Otherwise, there exists a countable ordinal ξsuch that $C(U_{n+1})$is isomorphic to $C(\xi)$. Consequently,

$$C(\xi, X) \sim C(U_{n+1}, X). \qquad (2.10)$$

Furthermore, it follows from (1.1)and (1.2)that

$$C([0, \xi^\omega] \times U_{n+2}) \sim C(U_{n+2}) \text{ and } C(\xi^\omega) \sim C([0, \xi^\omega] \times [0, \xi]).$$

Therefore

$$C(\xi^\omega, X) \sim C(\xi^\omega, C(\xi, X)) \sim C(\xi^\omega, C(U_{n+2}, X)) \sim C(U_{n+2}, X). \qquad (2.11)$$

Thus, by (2.10) and (2.11)we see that
$C(\xi, X) \sim C(\xi^\omega, X)$,
which contradicts Proposition2.7 and the theorem follows. □

3. On the isomorphic classification of $C(U_n, Y \oplus l_{(q-\varepsilon)}(\Gamma))$spaces,$1 < (q - \varepsilon) < \infty$

The purpose of this section is to provide the proof of Theorem 1.2. We shall denote by $\{e_{i,j}\}_{i,j=1}^\infty$ the canonical basis of $l_1(l_q)$, i.e.,

$$\left\|\sum_{i,j=1}^{\infty} a_{i,j} e_{i,j}\right\| = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|a_{i,j}|^q\right)^{1/q},$$

for i all $\{a_{i,j}\}_{i,j=1}^{\infty} \subseteq R$.

The next lemma is obtained by a gliding hump argument and a simple perturbation argument which are well-known, [11] but we include the proof for completeness.

Lemma 3.1. Let X be a Banach space and $1 < q < \infty$. Let T be a linear operator from $l_1(l_q)$to $X \oplus l_q$ and P^2 the natural contraction and projection from $X \oplus l_q$ onto l_q. Then:

(a)For all double sequences $\{\varepsilon_{i,j}\}_{i,j=1}^{\infty}$of positive numbers there exist a double sequence $\{b_{i,j}\}_{i,j=1}^{\infty} \subseteq l_q$ with pairwise disjoint finite supports and subsequences $N_j \subseteq N$ such that denoting $N_j = \{[i,j]\}_{i=1}^{\infty}$,

$$\|P^2\, T^2\,(e_{[i,j],j}) - b_{i,j}\| < \varepsilon_{i,j}\,, \qquad (3.1)$$

for every $1 \le i,j < \omega$.

(b)If T^2 is an into isomorphism then there exist subsequences $N_j = \{[i,j]\}_{i=1}^{\infty} \subseteq N$, $1 \le j < \omega$, and an isomorphism $(T^2)^{-1}$from the span of $\{e_{[i,j],j}\}_{i,j=1}^{\infty}$into $X \oplus l_q$ such that $\{P^2\, T^2(e_{[i,j],j})\}_{i,j=1}^{\infty}$is a double sequence in $l_q$with pairwise disjoint finite supports.

Proof. (a) Define an order $\prec$on $N \times N$ by $(i,j) \prec (k,l)$ if, and only if, $i + j < k + l$ or $i + j = k + l$ and $i < k$.

Assume we already found the initial segments of $N_j = \{[i,j]\}_{i=1}^{k_j}$for $(i,j) \prec (i_0, j_0)$. We need to find $[i_0, j_0]$and b_{i_0,j_0}. Since $\{e_{i,j_0}\}_{i=1}^{\infty}$tends weakly to zero, for $i_0$large enough $\|(P^2)_i T^2(e_{i_0,j_0})\| < \varepsilon_{i_0,j_0}/2$, where $(P^2)_i$ is the contraction and projection onto $(S^2)_i$, the finite union of the supports of $\{b_{ij}\}_{(i,j)\prec(i_0,j_0)}$.

Now, for all $1 \le n < \omega$, denote by $\mathbb{R}^n$ the natural projection of l_q given by $R_n(\{a_i\}_{i=1}^{\infty}) = (a_1, a_2, \ldots, a_n, 0, 0, \ldots)$.

Pick $1 \le m < \omega$ strictly greater than the maximum of Siand such that $\|(P^2)_i T^2(e_{i_0,j_0}) - R_m P^2 T^2(e_{i_0,j_0})\| < \varepsilon_{i_0,j_0}/2$.

So, it suffices to define

$$b_{i_0,j_0} = (R_m - (P^2)_i)P^2 T^2(e_{i_0,j_0}).$$

(b) Fix a double sequence $\{\varepsilon_{i,j}\}_{i,j=1}^{\infty}$of positive numbers such that $\sum_{i,j=1}^{\infty} \varepsilon_{ij}^p < 1/\|(T^2)^{-1}\|^p$, where $1/(q-\varepsilon) + 1/q = 1$. By the item (a) there exist subsequences $N_j = \{[i,j]\}_{i=1}^{\infty} \subseteq N$ and a double sequence $\{b_{ij}\}_{i,j=1}^{\infty} \subseteq l_q$ with pairwise disjoint finite supports and satisfying (3.1). Define the linear operator $\tilde{T}$from the span of $\{e_{[i,j],j}\}_{i,j=1}^{\infty}$ *to* $X \oplus l_q$ by

$$(T^2)^{-1}(e_{[i,j],j}) = (I^2 - P^2)T^2(e_{[i,j],j}) + b_{ij}.$$

Then $\|T^2 - (T^2)^{-1}(e_{[i,j],j})\| < \varepsilon_{i,j}$, for every $1 \le i,j < \omega$. Therefore $(T^2)^{-1}$ is an into isomorphism and $P^2(T^2)^{-1}(e_{[i,j],j}) = b_{ij}$, for every $1 \le i,j < \omega$. □

Proposition 3.2. Let X be a Banach space and $1 < q < \infty$. Suppose that $X \oplus l_q$ contains a copy of $l_1(l_q)$. Then Xcontains a copy of l_q.

Proof. Let T^2 be an isomorphism from $l_1(l_q)$into $X \oplus l_q$. Initially observe that for all infinite sequences $N_j \subseteq N$, $1 \le j < \omega$, $\{e_{ij}\}_{i,j=1, i\in N_j}^{\infty}$ spans in $l_1(l_q)$ a subspace isometric to $l_1(l_q)$.Thus, thanks to Lemma3.1 we may suppose that $\{P^2 T^2(e_{ij})\}_{i,j=1}^{\infty}$ is a sequence in $l_q$with pairwise disjoint finite supports.

First of all notice that for any finite set $A \subset N \times N$ and sequence $\{a_{n,j}\}_{n,j=1}^{\infty} \subseteq R$ we have

$$\left\|\sum_{(n,j)\in A} a_{n,j}\ P^2T^2(e_{n,j})\right\| \le M\left(\sum_{(n,j)\in A}|a_{n,j}|^q\right)^{1/q}, \qquad (3.2)$$

where $M^2 = \|P^2\|\|T^2\|$.

Now pick $0 < \varepsilon < 1$and $1 \le k < \omega$ satisfying $M^2\|(T^2)^{-1}\|k^{-1/(q-\varepsilon)} < \varepsilon$. Observe that for all $\{a_n\}_{n=1}^{\infty} \subseteq R$ and $0 \le \varepsilon < (\omega - 1)$ we have

$$\left\|\sum_{n=1}^{m} a_n\left(\frac{1}{k}\sum_{j=1}^{k} e_{n,j}\right)\right\| = \left(\sum_{n=1}^{m}|a_n|^q\right)^{\frac{1}{q}}, \quad (3.3)$$

that is, $\{k^{-1}\sum_{j=1}^{k} e_{n,j}\}_{n=1}^{\infty}$ is equivalent to the l_q basis. Denote by W be the span of these vectors.

Let $\sum_{n=1}^{m} a_n \left(\frac{1}{k}\sum_{j=1}^{k} T^2(e_{n,j})\right)$ be a vector of norm less than or equal to1. By (3.2)and (3.3)we infer

$$\left\|P^2\left(\sum_{n=1}^{m} a_n\left(\frac{1}{k}\sum_{j=1}^{k} T^2(e_{n,j})\right)\right)\right\| \le M^2\left(\sum_{n=1}^{m}\frac{|a_n|^q}{k^q}\right)^{\frac{1}{q}}$$

$$\le \frac{M^2}{k^{\frac{1}{(q-\varepsilon)}}}\left\|\sum_{n=1}^{m} a_n\left(\frac{1}{k}\sum_{j=1}^{k} e_{n,j}\right)\right\|$$

$$\le \frac{M^2}{k^{1/(q-\varepsilon)}}\|(T^2)^{-1}\|\left\|\sum_{n=1}^{m} a_n\left(\frac{1}{k}\sum_{j=1}^{k} T^2(e_{n,j})\right)\right\| \le \varepsilon$$

Consequently, if I^2 denotes the identity operator of $X \oplus l_q$, then $I^2 - P^2$ is an isomorphism from a subspace isomorphic to l_q into X.

Proof of theorem 1.2. The condition is of course sufficient. Let us show that it is also necessary. To do this, by Theorem4.2 it is enough to prove that $l_{(q-\varepsilon)}(\Gamma)$ is an $\alpha+\varepsilon$-quotient of $Y \oplus l_{(q-\varepsilon)}(\Gamma)$. Since $l_{(q-\varepsilon)}$ is a uniformly convex space and $(l_{(q-\varepsilon)}(\Gamma))^n \sim l_{(q-\varepsilon)}(\Gamma)$for every $1 \le n < \omega$, it suffices to prove that

$C(\xi, Y) \oplus l_{(q-\varepsilon)}(\Gamma) \nrightarrow c_0(l_{(q-\varepsilon)})$,

for every $\omega \le \xi < \alpha+\varepsilon$. But if this is not the case, then by duality and by the separability of $l_1(l_q)$ it follows that $l_1(Y^*) \oplus l_q$ contains a copy of $l_1(l_q)$.Thus, Proposition 3.2 impliesthat $l_1(Y^*)$ contains a copy of l_q. Then, by a standard gliding hump argument we can prove that Y^*contains a copy of l_q, see for instance, [3] a contradiction. This proves the theorem.

4. On the isomorphic classification sequence of $C(K, Y \oplus l_\infty(\Gamma))$spaces

In this section we prove Theorem1.3. First we need to state the following proposition.

Proposition 4.1. Let A and B be Banach spaces such that there exist a set Λ and $1 < (q-\varepsilon) < \infty$ satisfying

(a)$B \twoheadrightarrow l_{(q-\varepsilon)}(\Lambda)$,

(b)$B \nrightarrow c_0$,

(c) for any $\omega \le \xi < \alpha+\varepsilon$ and bounded linear operator $T: C(\xi, A) \to l_{(q-\varepsilon)}(\Lambda)$, we have dens$T(C(\xi, A)) < |\Lambda|$. Then $l_{(q-\varepsilon)}(\Lambda)$ is an $\alpha+\varepsilon$-quotient of $X = A \oplus B$.

Proof. Suppose that there exists a bounded linear operator T from $C(\xi, A) \oplus B^n$ onto $c_0(l_{(q-\varepsilon)}(\Lambda))$ for some $\omega \le \xi < \alpha+\varepsilon$ and $1 \le n < \omega$.

Given $1 \le m < \omega$, we will denote by $(P^2)_m$ the natural contraction and projection on $c_0(l_{(q-\varepsilon)}(\Lambda))$ onto the m-th coordinates, that is, $(P^2)_m: c_0(l_{(q-\varepsilon)}(\Lambda)) \to c_0(l_{(q-\varepsilon)}(\Lambda))$ defined by

$(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots) \to (0, 0, \ldots, x_m, 0, 0, \ldots)$.

By our hypothesis we deduce that dens $(P^2)_m T^2(C(\xi, A)) < |\Lambda|$, for every $1 \le m < \omega$. Hence there exists a subset $(\Lambda-\varepsilon)$ of Λ with $\varepsilon > 0$ such that $T^2(x)(\gamma)(m) = 0$ for every $x \in C(\xi, A), \gamma \notin (\Lambda-\varepsilon)$ and $1 \le m < \omega$. We identify in the natural way $c_0(l_{(q-\varepsilon)}(\Lambda-\varepsilon))$ as a subset of $c_0(l_{(q-\varepsilon)}(\Lambda))$. Let Qbe the natural contraction and projection from $c_0(l_{(q-\varepsilon)}(\Lambda))$ onto $c_0(l_{(q-\varepsilon)}(\Lambda-\varepsilon))$ So, it is easy to see that the following operator is onto

$Q^2 T^2{}_{|B^n}: B^n \to c_0(l_{(q-\varepsilon)}(\Lambda \setminus (\Lambda-\varepsilon)))$.

Consequently,

$$B^n \twoheadrightarrow c_0.$$

Thus, $c_0$is isomorphic to a quotient of B. This contradicts (b) and the proof is complete.

Proof of Theorem 1.3. Sufficiency is obvious. Let us see necessity. Notice that if $\Gamma$is an infinite set, then by [15] we have that $l_2(2^{|\Gamma|})$ is isomorphic to a quotient of $l_\infty(\Gamma)$. Moreover, by [6] it follows that $l_\infty(\Gamma)$ has no quotient isomorphic to c_0. So, by Proposition 4.1with $B = l_\infty(\Gamma)$ and $\Lambda = 2^{|\Gamma|}$, we deduce that $l_2(2^{|\Gamma|})$ is an $\alpha+\varepsilon$-quotient of $Y \oplus l_\infty(\Gamma)$. So, by Theorem 2.4. we are done.

5. On the isomorphic classification of Sequence of $C(U_n)$spaces

In this section show that $\alpha+\varepsilon$-quotient of Banach spaces can also be used to get the isomorphic classifications of certain sequence of $C(U_n)$ spaces for large compact sequences of Hausdorff spaces U_n. Let us start with a closely related result to Theorem 2.4.

Proposition 5.1. Let X be a Banach space having an $\alpha+\varepsilon$-quotient space Z which is uniformly convex. Write $X = A \oplus B$ as in Definition 2.1. Then for all infinite compact sequences of metric spaces U_{n+1} and U_{n+2},

$$A \oplus C(U_{n+1}, B) \sim A \oplus C(U_{n+2}, B) \Rightarrow C(U_{n+1}) \sim C(U_{n+2}).$$

Proof. We consider two cases:

Case 1. U_{n+1} and U_{n+2} are countable. Pick ξ and η infinite countable ordinals such that $C(U_{n+1})$ is isomorphic to $C(\xi)$ and $C(U_{n+2})$ is isomorphic to $C(\eta)$. Without loss of generality we may assume that $\xi \leq \eta$. Then,

$$A \oplus C(\xi, B) \sim A \oplus C(\eta, B) \twoheadrightarrow C(\eta, B) \twoheadrightarrow C(\eta, Z)$$

Hence by Proposition 2.5. and (1.2) we infer that $C(U_{n+1})$ is isomorphic to $C(U_{n+2})$.

Case 2. U_{n+2} is uncountable. We will show that $C(U_{n+1})$ is isomorphic to $C(U_{n+2})$ by proving that U_{n+1} is uncountable. Otherwise, there exists a countable ordinal ξ such that $C(U_{n+1})$ is isomorphic to $C(\xi)$. Thus,

$$A \oplus C(\xi, B) \sim A \oplus C(U_{n+2}, B) \twoheadrightarrow C(\xi^{\omega}, B) \twoheadrightarrow C(\xi^{\omega}, Z)$$

a contradiction by Proposition2.5. and the proof of proposition is complete.

Recall that a topological space S_n is said to be dispersed if every nonempty subset of S contains a relatively isolated point. Furthermore, the topological weight of a topological space U_n is the smallest cardinal m such that there exists a base of open subsets of U_n of cardinality m.

Theorem 5.2. Let Γ be an infinite set and S_n a dispersed compact sequences of Hausdorff space or an infinite compact sequences of Hausdorff space having topological weight strictly less than $2^{|\Gamma|}$. Then for any infinite compact sequences of metric spaces U_{n+1} and U_{n+2},

(a) $C(U_{n+1} \times (S_n \oplus (\alpha+\varepsilon_2)\Gamma)) \sim C(U_{n+2} \times (S_n \oplus (\alpha+\varepsilon_2)\Gamma)) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2})$.

(b) $C(S_n \oplus (U_{n+1} \times (\alpha+\varepsilon_2)\Gamma)) \sim C(S_n \oplus (U_{n+2} \times (\alpha+\varepsilon_2)\Gamma)) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2})$.

Proof. Of course, the condition $C(U_{n+1}) \sim C(U_{n+2})$ is sufficient for both statements of the proposition. We will show that this condition is also necessary. First of all observe that

$$C(U_n \times (S_n \oplus (\alpha+\varepsilon_2)\Gamma)) \sim C(U_n, C(S_n) \oplus l_\infty(\Gamma)),$$

and

$$C(S_n \oplus (U_n \times (\alpha+\varepsilon_2)\Gamma)) \sim C(S_n) \oplus C(U_n, l_\infty(\Gamma)),$$

for every compact sequences of Hausdorff space U_n.

Set $A = C(S_n), B = l_\infty(\Gamma)$ and $Z = l_2(2^{|\Gamma|})$. In view of Theorem2.4 and Proposition 5.1. it suffices to show that $l_2(2^{|\Gamma|})$ is an $\alpha+\varepsilon$-quotient of $X = C(S_n) \oplus l_\infty(\Gamma)$. We distinguish two cases:

Case 1. S is dispersed. We know that $l_2(2^{|\Gamma|})$ is isomorphic to a quotient of $l_\infty(\Gamma)$. On the other hand, notice that for any ordinal $\omega \leq \xi < \alpha + \varepsilon$ *the* compact space $[0,\xi] \times S$ is also dispersed. Moreover, it is well-known that any bounded liner operator T from $C([0,\xi] \times S_n)$ to $l_2(2^{|\Gamma|})$ is compact. [6,7] Therefore, by [18] $densT(C([0,\xi] \times S_n)) \leq \aleph_0 < 2^{|\Gamma|}$. Thus, it is enough to apply Proposition 4.1 with $\Lambda = 2^{|\Gamma|}$.

Case 2. The topological weight of S_n is strictly less than $2^{|\Gamma|}$. In this case, $densC(S_n) < 2^{|\Gamma|}$ [18] and by Proposition 4.1 with $\Lambda = 2^{|\Gamma|}$ we are done.

Theorem 5.3. Let Ω be an infinite Stonean space and S_n a dispersed compact sequences of Hausdorff space or an infinite compact sequences of Hausdorff space having topological weight strictly less than $2^{\aleph_0}$. Then for any infinite compact sequences of metric spaces U_{n+1} and U_{n+2},

(a) $C(U_{n+1} \times (S_n \oplus \Omega)) \sim C(U_{n+2} \times (S_n \oplus \Omega)) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2})$.
(b) $C(S_n \oplus (U_{n+1} \times \Omega)) \sim C(S_n \oplus (U_{n+2} \times \Omega)) \Leftrightarrow C(U_{n+1}) \sim C(U_{n+2})$.

Proof. Let us show the non-trivial implications. By [6] $C(\Omega)$ has a quotient isomorphic to l_∞. More-over, l_∞ has a quotient isomorphic to $l_2(2^{\aleph_0})$. So, it is enough to proceed as in the proof of Theorem 5.2

Remark 5.4. Regarding the statements of Theorem 2.4 and Proposition 5.1 observe that if $X = A \oplus B$ as in Definition 2.1, then we do not have necessarily

$C(U_n, X) \sim A \oplus C(U_n, B)$,

for every infinite compact sequences of metric space U_n.

Indeed, on the one hand by Proposition 3.2. and [1] we deduce that the $l_{(q-\varepsilon)}$ space with $1 < (q-\varepsilon) < 2$ is an $\alpha+\varepsilon$-quotient of $X = l_\infty \oplus l_{(q-\varepsilon)}$.

On the other hand, since $l_2(2^{\aleph_0})$is a quotient of l_∞, we conclude by Proposition 4.1that $l_2(2^{\aleph_0})$is an $\alpha+\varepsilon$-quotient of $l_{(q-\varepsilon)} \oplus l_\infty$. So, by the item (c) of Definition 2.1. we infer

$$C(U_n, X) \sim A \oplus C(U_n, B), C(\omega, l_{(q-\varepsilon)}) \oplus l_\infty \nrightarrow C(\omega, l_2(2^{\aleph_0})). \quad (5.1)$$

Consequently, we cannot have

$C(\omega, l_\infty \oplus l_{(q-\varepsilon)}) \sim l_\infty \oplus C(\omega, l_{(q-\varepsilon)})$,

otherwise,

$C(\omega, l_{(q-\varepsilon)}) \oplus l_\infty \sim C(\omega, l_\infty \oplus l_{(q-\varepsilon)}) \twoheadrightarrow C(\omega, l_\infty) \twoheadrightarrow C(\omega, l_2(2^{\aleph_0}))$,

a contradiction by (5.1) (See in [19]).

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