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*Research Paper*

# Renorming of Banach Spaces with MIP and MIP*

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## ABSTRACT

In this paper, we renorm the Banach spaces with Mazur intersection property (MIP) or dual space with w*-Mazur intersection property (MIP*).Some basic results concerning the MIP and MIP*are introduced and study nearly about the problems 5.1 and 5.3 mentioned below. Additional conditions to these problems are found to have positive answer. However these problems have negative answer in general.

***Key words:*** Asplund Space, Mazur Space, Slice, Denting Point, Frechet Derivative of a Norm, Exposed point.

## 0. INTRODUCTION

The MIP is a purely geometrical property and well studied notion in geometry of Banach space. In 1933, it was S. Mazur who first considers the following smoothness property in normed linear spaces, called the Mazur Intersection Property (MIP); every bounded closed convex set can be represented as an intersection of closed balls, i.e. it is a ball covering property. [10,18] He started his investigation to determine those normed spaces which possess MIP [2] and has since been studied extensively over the years. [17] He showed that any reflexive Banach space with a Frechet-differentiable norm has this property. [22]

Later, a systematic study of this topic was initiated by R. R. Phelps. He continued his investigation, characterizing finite dimensional space with MIP. [6,7,22] Nearly two decades later, Phelps' results were extended by J. R. Giles, D. A. Gregory and B. Sims to general normed linear spaces developing an idea due to F. Sullivan. [8] In 1978, they gave some characterizations of this property. [6] They showed in [13] that a Banach space X has MIP if and only if the set of w*- denting points of B(X*) is norm dense in S(X*). They also showed that in dual Banach spaces, the MIP implies reflexivity and considered the weaker property that every weak* compact convex set in a dual space is the intersection of balls B(X*). [2,6]

They raised the question whether every Banach space with the MIP is an Asplund space. They also characterized the associated property for a dual space, called the weak* Mazur intersection property: every bounded weak* closed convex set can be represented as an intersection of closed dual balls. Associated with MIP, we have also the concepts of Mazur sets and Mazur spaces which characterize smooth spaces this characterization is given by F. Sullivan in [8] It has been recently proved by Jimenez and Moreno [25] that Kumen space is an Asplund space with no equivalent norm with MIP.

MIP is the Euclidean space property and can be regarded as a separation property by a ball which is stronger than the classical separation property by hyper planes. [2] A major theme in the study of Banach spaces is to determine classes of those spaces where particular Euclidean space properties hold. [27] It is weaker than Frechet differentiability of the norm and gave a dual characterization for MIP, in a finite dimensional case which is extended to infinite dimensional case where the MIP dually characterized .This fact was proved by Phelps. [1] Both MIP and MIP* are metric properties and hence invariant under isometries but not under isomorphisms. Recently Sullivan in [8] has given characterization of smooth spaces with MIP. After these pioneering works, many authors directed their intention to the systematic study of Banach spaces that admit MIP and w*- MIP. Some discussions on MIP-related renorming questions may be found in [1,19 -21]

**Notations and Terminologies**

In the sequel, (X, ‖· ‖) is the real Banach space with norm ‖·‖; S(X) is the unit sphere of X; (X*, ‖·‖*) is the dual space of X; S(X*) is the unit sphere of X*; B(X) is the unit ball of in X; B(X*) is the unit ball of X*. The short forms MIP, MIP*, LUR, FD, CCB, WCG stand for Mazur intersection property, w*- Mazur intersection property, locally uniform rotund norm, Frechet differentiable norm, and closed convex bounded set weakly compactly generated space respectively.

## 1. Some Definitions

**1.1** We say that a Banach space X has **MIP** if every closed convex and bounded set C in X is the intersection of closed balls containing it, i. e. every such set can be expressed as an intersection of the closed balls . More Precisely, $C = \cap_\alpha B(x_\alpha, \rho_\alpha)$ for some system of points $x_\alpha \in X$, and radii $\rho_\alpha$. [11, 18, 3,28]

**1.2** A closed, convex and bounded set *C* is an intersection of balls if it satisfies the separation property: for every $x \notin C$, there is a closed ball B such that B ⊂ C but $x \notin$ B. This property can be strengthening by replacing *x* by hyperplane. Hence, the MIP can be regarded as a separation property by a ball which is stronger than the classical separation property by hyper planes. [2] This easy and useful fact will be used extensively throughout the rest of the paper.

**1.3 Continuum hypothesis (CH)**: The conjuncture made by George Cantor that there is no set with cardinal number between $\aleph_0$ which is a set of natural numbers and cardinal number of set of real numbers, i. e. continuum. By CH we remark that MIP has connections with other parts of Banach space theory, such as the ball topology etc.

**1.4 Kumen Shelah Space.** It is a non separable Asplund space satisfying that for every uncountable set $\{x_i\}_{i\in I}$ in the space there is $i_0$ such that

$$x_{i_0} \in \overline{\text{con}}\left(\{x_{i_0}\}_{I-\{i_0\}}\right) \quad [2]$$

**1.5** Recall that a biorthogonal system $\{x_i, x_i^*\}_{i\in I} \subset X\times X^*$ is fundamental provided

$$X = \overline{\text{span}}\left(\{x_i\}_{i\in I}\right).$$

**1.6**. A norm ‖.‖ on a Banach space X is Frechet differentiable at $x_0 \in X$ if, for every $h \in X$ ,we have

$$\|x_0 + h\| - \|x_0\| - \|.\|'(x_0)\, h = r(h),$$

where $\lim_{h\to 0} \frac{r(h)}{\|h\|} = 0.$

where $f_0 = \|.\|'(x_0)$ is Frechet derivative norm of ‖· ‖ at $x_0$.

**1.7** By **Bio-Shop Phelps theorem** we can find f ∈ X* and $x_0 \in C$ such that $f(x_0) = \sup_{x\in C} f(x)$ .

**1.8** A Banach space X is said to be weakly compactly generated (WCG) if there is weakly compact set K in X such that X = $\overline{span}$ (K ) i.e. K generates X if X = $\overline{span}$ (K) , i.e . X contains weakly compact total subset, or K is dense in X.

## 2. Some Renorming Results

The question whether a Banach space can be renormed with MIP or a dual space with MIP* has not an easy answer. So let us gather some renorming results.

**Theorem 2.1** If a norm ‖.‖ on a Banach space X is Frechet differentiable, then (X,‖.‖) satisfies MIP. Its **proof is**

**standard** and can be seen in [[2, 7, 13, 14], p. 56, [15], p. 397, [1,28]].

**Theorem 2.1** implies that a norm with Frechet differentiable norm has MIP but it seems to be an open question whether a norm with the Mazur intersection property has some point of Frechet differentiability? [16] Indeed, every Frechet differentiable norm has the Mazur intersection property, the Kumen space is a non separable Asplund space admitting no Frechet differentiable norms.

**Theorem 2.2** Mazur space with MIP admits an equivalent Frechet differentiable norm [10,28]

**Theorem 2.3** For a Banach space X with MIP implies X** has the MIP*[[13, 22], p.22].

**Theorem 2.4** (Phelps) for a separable Banach space X, X can be renormed to have the Mazur intersection property if and only if X* is separable. [28]

**Example 2.5** Any WCG Banach space with WCG dual Banach space can be renormed to have MIP and MIP* .We note that such a space X can equivalently renormed so that both X and X* are LUR. Because WCG spaces are among those that can be LUR-renormed studied by Troyanski, Modulo Amir and Lindenstrauss.

## 3. Some Properties of MIP and MIP*

1. The three space property is satisfied by MIP and MIP* [16,22]
2. Every Banach space embeds isometrically into a Banach space with MIP answering question asked by Giles, Gregory and Sims. [2]
3. We note that no counterexample is possible in finite dimensions as smoothness is hereditary and implies the MIP in finite dimensions.
4. MIP is not stable under subspaces.
5. Existence of equivalent norm with MIP is hereditary.
6. A family of spaces admits an equivalent norm with MIP is stable under $C_0$ and $l_p$ sums. [1]
7. All the known examples where the heredity of the MIP fails are non-smooth spaces. So, it appropriate to ask: Is the property of being a smooth space with the MIP hereditary?
8. Suppose X can be renormed to have MIP. Then dens X = dens X* [28]

## 4. Slices, denting points, strongly exposed points and their w*-star forms.

It is worth to mention the notion of above geometric properties of a Banach space because these can be used to characterize MIP and MIP*.

**4.1**. A **slice of the ball B (X)** determined by $f \in S(X^*)$ and $\delta > 0$ is a subset of B(X) of the form $S(B(X), f, \delta) = \{ x \in B(X) : f(x) > r - \delta \}$ for some $0 < \delta < r$, r is the radius of ball.

A slice of the ball **B(X*)** determined by $\tilde{x} \in B(\widetilde{X})$ is called w*-slice of the ball B(X*). [24] Let $A \subset X$ a point $x \in A$ is said to be exposed by $f \in X^*$ at $x$ if f attains its maximum at $x$ and no other point. Exposed point of A implies extreme point of A but not conversely. A point $x \in A$ is strongly exposed by $f \in X^*$ if in addition every sequence $(x_n)$ in A which satisfies $f(x_n) \to f(x)$ implies $x_n \to x$.

**4.2** .We say that $\boldsymbol{x} \in S(X)$ is called denting point of the ball B(X) if for all $\epsilon > 0$, $x$ is contained in the slice of B(X) of diameter less than $\epsilon > 0$, $f \in S(X^*)$ is called **w*-denting point of** B(X*) if for all $\epsilon > 0$, f is contained in the slice of B(X*) of diameter less than $\epsilon > 0$. A point $x_0$ is denting point if for all $\epsilon > 0$ $x \notin \overline{con}(K - B(x_0, \epsilon))$.

Equivalently , the point $x \in A \subset X$ is said to be a denting point if for every $\epsilon > 0$ there exist $f \in X^*$ and $\alpha > 0$ such that $x \in S(A, f, \alpha) \subset B(x, y)$. If in this definition f does not depend on y, then the point $x$ is said to be a strongly exposed point and the functional f is said to be strongly exposing functional.

**4.3.** We say that $x \in S(X)$ is **strongly exposed point** of B (X) if there is $f \in S(X^*)$ such that for all $\epsilon > 0$ , $0 < r < \delta$ for which $\boldsymbol{x} \in S(B(X), f, \delta)$ and diam. $S(B(X), f, \delta) < \epsilon$

**4.4** .We say that $f \in S(X^*)$ is **w*-strongly exposed point** of B(X*) if there is $\tilde{x} \in S(\widetilde{X})$ strongly exposes B(X*) at f.

**4 .5** .**The set of w*-** denting point of B(X*) is norm dense in S(X*). Is the set of **w*-** exposed point of B(X*) is norm dense in S(X*)?

If the norm of X is Frechet differentiable on a norm dense set in S(X) then above question has positive answer. This is the case when X is an Asplund space. So all reflexive and separable spaces with MIP satisfies this condition. [6]

a. Every denting and strongly exposed point is extreme point [30]
b. Extreme point is not strongly exposed point [30]
c. Strongly exposed point is denting point but denting point is not strongly exposed point even in finite dimensional space. [30] Is set of strongly exposed point of B(X) is norm dense in S(X)?

Since every strongly exposed point is denting point and set of set of denting point of B(X) is norm dense in S(X) so above question has positive answer.

**With these concepts the following results are proved**.

**Theorem 4.6** [7] **Let X be a normed space then**

a) X has the MIP if the set of w*-strongly exposed points of the unit ball B(X*) of the dual X* is norm dense in the unit sphere S(X*).
b) If X has the MIP, every support mapping on X maps norm dense sub sets of S(X) to norm dense subsets of S(X*).
c) A finite dimensional normed linear space X has the MIP if and only if the extreme points of B(X*) are norm dense in S(X*) [2,7] Phelps also asked whether the sufficient condition (a) is also necessary. To date, this remains an open question. Those Banach spaces which have MIP and whose dual have MIP* do have strongly exposed point property and such spaces having the following characterization.

**Theorem 4.7** If a Banach space X has MIP then X* has MIP* if and only if the set of strongly exposed point of B(X) is dense in S(X) and set of w*-strongly exposed point of B(X*) is dense in S(X*).

**We have an open question:** If a Banach space has MIP, does there exist any w*-strongly exposed point on B(X*)?

**Indeed** ,we have the positive answer under the following conditions ; if X has the MIP and the norm is Frechet differentiable on a dense subset of S(X), then the w*-strongly exposed points of B(X*) are dense in S(X*). In particular, this happens when X has the MIP and either X* has the w*-MIP or X is Asplund. [13]

**Theorem 4.8** For a normed linear space X, the following are equivalent:

a) The w*-denting points of B(X*) are norm dense in S(X).
b) X has the MIP.
c) Every support mapping on X maps norm dense subsets of S(X) to norm dense subsets of S(X*).

**Theorem 4.9** Let X be a Banach space. Then the following are equivalent.

a) X has the Mazur intersection property. [6]
b) For every $\epsilon > 0$ there is a norm dense set in S(X*) each of whose points is in a w*-slice of B(X*) having diameter less than $\epsilon$.
c) The w*-denting points of B(X*) are norm dense in S(X*).

**Remark 4.10** X* has MIP* if and only if the set of denting point of B(X) is norm dense in S(X).

Phelps theorem was extended to the infinite dimensional case given in [6], where separation property was dually characterized.

Notice that if X is separable and has the MIP, Phelp's condition (b) (or, Theorem 4.9(c) above) implies that it has a separable dual and hence is Asplund.

**Theorem 4.10**.The Kumen and Shelah spaces do not admit an equivalent norm with the Mazur intersection property. Analogously, the duals of the previous spaces do not admit a dual norm with the MIP* [2]

**Corollary 4.11** A finite dimensional normed linear space X has the MIP if and only if the set of extreme points of B(X*) is dense in S(X*).

**Theorem 4.12** K. S. Lau had shown that a reflexive Banach Space has the Mazur intersection property if and only if every closed convex bounded set is the closed convex hull of its farthest point. [4]

**Example 4.13 .**There is a space X with a Frechet differentiable norm, and hence with MIP, and a closed bounded convex set K in X that lacks farthest points.

**Proof:** Recall that if the norm is strictly convex (respectively, locally uniformly convex), any farthest point of a closed bounded convex set is also an extreme (resp. denting) point. So, if every closed bounded convex set admits farthest points, then the space must necessarily have the Krein-Milman Property (KMP) (resp. the RNP) (see [4] or [7] for details). However, the space $c_0$ , which does not have the KMP, admits a strictly convex Frechet differentiable norm (seee.g., [4, 6]).

**Example 4.14**.The space $c_0$ has a strictly convex Frechet differentiable renorming [3, Theorem 7.1 (ii)] which, thus, has the MIP. However, since the unit ball of the usual norm on $c_0$ lacks extreme points, it must lack farthest points in the new norm. [23]

**Theorem 4.15** The Banach space $C_0$ (T) admits a norm with the Mazur intersection property whenever T is a tree space [[14], Theo.7.1.[1],[2]].

**Theorem 4 .16** Let T: X→ Y be a linear operator such that T and T* are injective. If Y has an equivalent norm with MIP, then X has an equivalent norm with MIP. [1]

**Theorem 4.17** Let X* be a dual Banach space with a fundamental biorthogonal system $\{(x^*_\alpha , x_\alpha )\}_{\alpha < \omega 1} \subset X^{*} \times X^{**}$, with the property that $x_\alpha \in X \times X^{**}$.Then X admits an equivalent norm with the Mazur intersection property. [17,28]

**Theorem 4.18** A finite dimensional normed linear space has MIP if and only if the set of extreme point of B(X*) dense in S(X*). In particular, a 2-dimensional space has the MIP if and only if it is smooth. [2]

**Theorem 4.19** For a reflexive Banach spaces X with MIP ,a bounded sequence $\{x_n\}$ converges weakly *to* $x$ if and only if for all closed ball containing a subsequence also contains $x$. [2,6]

**Theorem 4.20** (Phelps) for a separable Banach space X, X can be renormed to have MIP if and only if X* is separable. [28]

## 5. Mazur intersection property and Asplund spaces

We remark that MIP has connections with other parts of Banach space theory, such as Asplund space; a Banach space X is an Asplund space if every separable subspace of X has a separable dual space. [17] In this section, we see how two important problems about MIP are answered in the negative which remain open in the past posed by J. R. Giles, D. A. Gregory and B. Sims, on the Mazur intersection property, by exhibiting a class of non Asplund spaces admitting an equivalent norm with the above property. On the other hand, if the continuum hypothesis is assumed, we find an Asplund space admitting no equivalent norm with the Mazur intersection property and whose dual space admits no equivalent norm with the weak* Mazur intersection property. [16] It was for long time an open problem to determine whether spaces with the MIP are Asplund spaces. Also, it was unknown if every Asplund space admits a norm with the MIP or, in particular, a Frechet differentiable norm. The latter was shown in the negative by Haydon [27] First and second problems were also answered in the negative in [25] and we study like this:

**Problem 5 .1 Is every Banach with MIP an Asplund space? [6]**

The answer is **No** because there exists a Banach spaces with the Mazur intersection property which are not Asplund spaces, i.e. Mazur intersection property does not characterize Asplund spaces. [28] A Banach space with MIP* is not necessarily weak* Asplund for example $l_1$ is not w*-Asplund. [2,6] Consequently, Mazur intersection property does not imply for a Banach space to be Asplund. Moreover, Banach spaces with the Mazur intersection property and whose dual have the weak*-

Mazur intersection property are not necessarily w*- Asplund spaces. [27]

**However, on adding some conditions to the above question we get the positive answer.**

**Theorem 5.2** [22, p.22] For a Banach space X.

a) If , the norm is Frechet differentiable at all $x \in$ S(X), then X has the MIP and is Asplund

b) For a separable Banach space X, X can be renormed to have the Mazur intersection property if and only if X is Asplund. [7,22]

c) (Bacak and H. 2008) Under Martin's Maximum axiom MM, every Asplund space of density character $\omega_1$ can be renormed to have the Mazur intersection property. [28]

d) In context problem 5.1, Sullivan [8] has shown that if for some ϵ > 0, $M_\epsilon$ = S(X), then X is Asplund.

**But the above question has negative answer in general.**

**Problem 5. 3. Does every Asplund space admit an equivalent norm with MIP?**

a) **Very recently,** Todercevic Lopaz-Abad proved under an axiom,that there is an Asplund spaces with RNP that has no norm with MIP ( see [HMVZ] within [14] ) .

b) An example of an Asplund space with MIP but having no Frechet renorming is not known. [28]

c) (CH) The space C(K), K the Kunen compact space, is an Asplund space of density character $\omega_1$ that cannot be renormed to have the Mazur intersection property(see [JiMo97]in [28] ), [5]

d) If the continuum hypothesis is assumed, There exits (at least under CH) an Asplund space admitting no equivalent norm with the Mazur intersection property, namely the Kunen space K [13, p. 1128] .Moreover, its dual space admits no equivalent norm with MIP* [16]

**Example 5.4** .The space X = C(K) of continuous functions defined on a scattered compact K (constructed by Kunen under CH) is a non separable Asplund space and has no equivalent norm with the Mazur intersection property .

**However on adding some conditions on an Asplund space the above question has positive answer.**

**Theorem 5.5** [22, p.22] For a Banach space X, if X has the MIP and the norm is Frechet differentiable on a dense subset of S(X), then the w*-strongly exposed points of B(X*) are dense in S(X*). In particular, this happens when X has the MIP and either X* has the MIP* or X is Asplund. [20]

**Theorem 5.6** [16] Let X* be a dual Banach space with a fundamental biorthogonal system {***x***i, *f*i}i∈ I ⊂ X × X*. Then, X admits an equivalent norm with the Mazur intersection property.

It is worth observing that, using **Theorem 3.6**, we can show that the example exhibited by Haydon (the set of continuous functions over the full uncountable branching tree of height ω1) in, [8] of an Asplund space without Frechet differentiable norms, can be equivalently renormed to have the Mazur intersection property.

**Theorem 5.7 Under** Martin's Maximum axiom MM, every Asplund space of density character $\omega_1$ has a renorming with the Mazur intersection property. [17,28]

Bandyopadaya and Roy [3,13,24] proved that if μ denotes Lebesgue measure in [0,1] and 1< p < ∞ then $L_p(\mu$, X) has MIP iff X has the MIP and is an Asplund space.

But in 1995, M .J .Sevilla and J. P. Moreno [11,12] has exhibited a class of non - Asplund spaces that admit an equivalent norm with the MIP.

**But the above question has negative answer in general.**

## 6. Mazur sets and Mazur spaces

Let $\mathcal{H}$ the family of all bounded closed convex subsets, $\mathcal{M}$ the family of all intersections of closed balls, the most interesting classes of convex sets and $\mathcal{P}$ the family of all Mazur sets. [27] About 75 years ago the study of the Banach spaces satisfying the property $\mathcal{M} = \mathcal{H}$ in a real Banach space X appears in several questions related to the convex geometry, fixed point theory and geometry of Banach spaces. The question of whether every closed, convex and bounded set of a normed space is in $\mathcal{M}$, a property which is known under the name of the *Mazur intersection property* (MIP). It received later renewed attention by Phelps, Giles and Bor-Luh among many other authors. However, some basic properties of $\mathcal{M}$ when the space fails the MIP have seldom been investigated [29and references therein] and some properties when $\mathcal{M} \neq \mathcal{H}$ were open. For example when $\mathcal{H}$ is equipped with the usual Hausdorff metric, is it true or false that $\mathcal{M}$ is (topologically) closed? (Answer is negative). [2,26]

A closed, convex and bounded set C is a *Mazur set* provided that for every hyperplane *H* such that dist(C, *H)* > 0, there is a ball *D* satisfying $C \subset D$ and $D \cap C = \phi$. In virtue of the Hahn-Banach theorem, Mazur sets are intersections of balls which simply satisfy a stronger separation property. When every intersection of balls is a Mazur set, we say that the space is a **Mazur space**. This class was introduced in, [27] where some of its structural properties were investigated.

We say that a set C defined in the normed space with MIP is a **Mazur set** if given any hyper plane H with d ( C, H) > 0 there is a ball D such that $C \subset D$ and supf (D) < $\lambda$ [2, 9]

Equivalently, We say that a set C is **Mazur set** if , given $f \in X^*$ with sup f(C) < λ then there exists a ball D such that $C \subseteq D$ and sup f (D) < λ .

Let $\mathcal{M}$ be the collection of all intersections of balls .Then spaces in which every element of $\mathcal{M}$ is a Mazur set ($\mathcal{M} = \wp$) is called **Mazur space.** [9] S. Mazur [2] showed that any reflexive Banach space with Frechet differentiable norm is always Mazur space. However, spaces with Frechet differentiable norm need not be Mazur space. However part in above is seen to be true by the following counter examples , the spaces $c_0(I)$ and $C_\infty(I)$ , where I is discrete set, are Mazur spaces with their separable Banach spaces with their usual norms. But they are not reflexive spaces.

**Theorem 6.1** For every set I, the space $(c_0(I); \|\cdot\|_\infty)$ is a Mazur space. [2]

**Example 6.2** one and two dimensional spaces are Mazur spaces but $(\mathbb{R}_3, \|\cdot\|_1)$ is not Mazur space because it is not stable under vector sum.

**Theorem 6.3** The set $\mathcal{M}$ is not stable under vector sums in $(\mathbb{R}^n, \|\cdot\|_1)$, n > 3 or in $l_1(I)$.

**Theorem 6.4** A Mazur space X satisfies the Mazur intersection property if and only if every norm one functional in X* is a weak* denting point of B (X*). [2,15]

**Corollary 6.5 A** Banach space has dimension less than three if and only if it is a Mazur space with respect to every equivalent norm. [2]

**Remark 6.6** Finite dimensional Banach spaces with MIP are not **Mazur space.** [15]

**Theorem 6.7** Every reflexive space with Frechet differentiable norm is Mazure space and every Mazur space with MIP is a smooth space. [2,9,18,22] However, spaces with Frechet differentiable norms need not be Mazur spaces, the latter fact is proved in. [25]

**Proof** In a reflexive space with a Frechet differentiable norm, every norm one functional of the dual is the differential of the norm at some point. Consequently, it is a weak*- strongly exposed point (and thus a weak* denting point) of the dual unit ball. On the other hand, it is well known that there is only a partial duality between smoothness and convexity. As a matter of fact, from the pioneering results about renormings on spaces of continuous functions on scattered compact spaces due to Talagr and, we know that there are spaces with Frechet differentiable norm whose dual space admits no rotund norm. This is the case, for instance, for *C* $([0; \omega_1])$. Since every weak*denting point is also an extreme point the proposition above implies that the dual norm of a Frechet norm in a Mazur space must be rotund. As a consequence, *C* $([0; \omega_1])$, endowed with equivalent Frechet differentiable norm is not a Mazur space.

The previous proposition shows that, in particular, a Mazur space with the MIP has a dual rotund norm and thus the norm of the space itself is Gateaux differentiable. □

**Theorem 6.8** (see reference 29 within [10]) A Mazur space with the MIP admits an equivalent Frechet differentiable norm.

**Theorem 6.9 (**see reference 27 within [2,10]) Any space with Frechet differentiable norm satisfies the MIP. Also every space whose dual satisfies the MIP is reflexive and each reflexive space with Frechet differentiable norm is a Mazur space. Finally Mazur spaces with the MIP are Asplund and Gateaux differentiable.

## CONCLUSION

The MIP has been an active topic of research for several years. The basic facts about MIP and MIP* with application to renorming theorems are seen. Some of its more remifications are discussed. Smooth spaces can be characterized with the use notion of MIP and Mazur spaces. We interplay Mazur set, Mazur space and MIP. So this topic is a fertile field in the garden of geometry of Banach space theory.

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