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Short Communication

On The Negative Pell Equation $y^2 = 35x^2 - 19$

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ABSTRACT

The negative Pell equation represented by the binary quadratic equation $y^2 = 35x^2 - 19$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented. Employing the solutions of the equation under consideration, the integer solutions for a few choices of hyperbola and parabola are obtained.

Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.
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INTRODUCTION

Diophantine equation of the form $y^2 = Dx^2 + 1$ where D is a given positive square-free integer, is known as Pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over the world, since antiquity, J.L. Lagrange proved that the positive Pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions where as the negative Pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In, [1] an elementary proof of a criterion for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solutions, whereas $y^2 = 65x^2 - 1, y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer. [2-9] More specifically, one may refer " The On-line Encyclopedia of integer sequences" (A031396, A130226, A031398) for values of D for which the negative Pell equation $y^2 = Dx^2 - 1$ is solvable or not.

In this communication, the negative Pell equation given by $y^2 = 35x^2 - 19$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

METHODS OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 35x^2 - 19 \quad (1)$$

whose smallest positive integer solution is

$$x_0 = 1,\ y_0 = 4$$

To obtain the other solutions of (1), consider the Pell equation $y^2 = 35x^2 + 1$ whose general solution is given by

$$\tilde{x}_s = \frac{1}{2\sqrt{35}} g_s, \tilde{y}_s = \frac{1}{2} f_s$$

Where

$$f_s = (6+\sqrt{35})^{s+1} + (6-\sqrt{35})^{s+1}$$

$$g_s = (6+\sqrt{35})^{s+1} - (6-\sqrt{35})^{s+1}, \quad s = 0, 1, 2, 3, ...$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_s, \tilde{y}_s)$ the other integer solutions of (1) are given by

$$2x_{s+1} = f_s + \frac{4}{\sqrt{35}} g_s,$$

$$2y_{s+1} = 4f_s + \sqrt{35} g_s$$

The recurrence relations satisfied by x and y are given by

$$y_{s+3} - 12y_{s+2} + y_{s+1} = 0, y_0 = 4, y_1 = 59$$

$$x_{s+3} - 12x_{s+2} + x_{s+1} = 0, x_0 = 1, x_1 = 10$$

Some numerical examples of x and y satisfying (1) are given in the following table

Table 1: Numerical Examples

s	x_s	y_s
0	1	4
1	10	59
2	119	704
3	1418	8389
4	16897	99964
5	201346	1191179
6	2399255	14194184
7	28589714	169139029

From the above table we observe some interesting relations among the solutions which are presented below:

1. x_s is alternatively odd and even
2. y_s is alternatively even and odd
3. $x_{2s-1} \equiv 0 \pmod 2$
4. $y_{2s} \equiv 0 \pmod 4$

A few interesting properties between the solutions and special numbers are given below:

1. $\frac{6}{19}[70x_{2s+2} - 8y_{2s+2} + 38]$ is a nasty number
2. $\frac{1}{19}[70x_{3s+2} - 8y_{3s+3}] + 3[(6+\sqrt{35})^{s+1} + (6-\sqrt{35})^{s+1}]$ is a cubical integer
3. Each of the following properties represents the perfect square

- $\frac{1}{19}[70x_{2s+2} - 8y_{2s+2}] + 2$
- $\frac{1}{19}[118x_{2s+2} - 8x_{2s+3}] + 2$
- $\frac{1}{57}[352x_{2s+2} - 2x_{2s+4}] + 2$
- $\frac{\sqrt{35}}{19}[2\sqrt{35}x_{2s+2} - \frac{8}{\sqrt{35}} y_{2s+2}] + 2$
- $\frac{\sqrt{35}}{57}[10\sqrt{35}x_{2s+2} - \frac{2}{\sqrt{35}} y_{2s+3}] + 2$
- $\frac{\sqrt{35}}{1349}[238\sqrt{35}x_{2s+2} - \frac{8}{\sqrt{35}} y_{2s+4}] + 2$
- $\frac{1}{19}[1408x_{2s+3} - 118x_{2s+4}] + 2$
- $\frac{\sqrt{35}}{57}[\sqrt{35}x_{2s+3} - \frac{59}{\sqrt{35}} y_{2s+2}] + 2$
- $\frac{\sqrt{35}}{19}[20\sqrt{35}x_{2s+3} - \frac{118}{\sqrt{35}} y_{2s+3}] + 2$
- $\frac{\sqrt{35}}{57}[119\sqrt{35}x_{2s+3} - \frac{59}{\sqrt{35}} y_{2s+4}] + 2$

- $\frac{\sqrt{35}}{1349}[2\sqrt{35}x_{2s+4}-\frac{1408}{\sqrt{35}}y_{2s+2}]+2$
- $\frac{\sqrt{35}}{57}[10\sqrt{35}x_{2s+4}-\frac{704}{\sqrt{35}}y_{2s+3}]+2$
- $\frac{\sqrt{35}}{19}[238\sqrt{35}x_{2s+4}-\frac{1408}{\sqrt{35}}y_{2s+4}]+2$
- $\frac{1}{19}[2y_{2s+3}-20y_{2s+2}]+2$
- $\frac{1}{114}[y_{2s+4}-119y_{2s+2}]+2$
- $\frac{1}{19}[20y_{2s+4}-238y_{2s+3}]+2$

4. $38x_{s+3}=456x_{s+2}-38x_{s+1}$
5. $38y_{s+1}=38x_{s+2}-228x_{s+1}$
6. $38y_{s+3}=228x_{s+2}-38x_{s+1}$
7. $38y_{s+3}=2698x_{s+2}-228x_{s+1}$
8. $1444x_{s+3}-17328y_{s+1}=102524x_{s+1}$
9. $8664x_{s+3}-1444y_{s+1}=102524x_{s+2}$
10. $38x_{s+3}x_{s+1}-456y_{s+1}x_{s+1}=2698x_{s+1}^2$
11. $8664x_{s+3}x_{s+2}-1444y_{s+1}x_{s+2}=102524x_{s+2}^2$
12. $8664x_{s+3}-17328y_{s+2}=8664x_{s+1}$
13. $38x_{s+3}-38y_{s+2}=228x_{s+2}$
14. $38x_{s+3}x_{s+1}-76y_{s+2}x_{s+1}=38x_{s+1}^2$
15. $38x_{s+3}x_{s+2}-38y_{s+2}x_{s+2}=228x_{s+2}^2$
16. $102524x_{s+3}-17328y_{s+3}=1444x_{s+1}$
17. $8664x_{s+3}-1444y_{s+3}=1444x_{s+2}$
18. $102524x_{s+3}x_{s+1}-17328y_{s+3}x_{s+1}=1444x_{s+1}^2$
19. $228x_{s+3}x_{s+2}-38y_{s+3}x_{s+2}=38x_{s+2}^2$
20. $1444y_{s+2}-8664y_{s+1}=50540x_{s+1}$
21. $8664y_{s+2}-1444y_{s+1}=50540x_{s+2}$
22. $38y_{s+2}x_{s+1}-228y_{s+1}x_{s+1}=1330x_{s+1}^2$
23. $228y_{s+2}x_{s+2}-38y_{s+1}x_{s+2}=1330x_{s+2}^2$
24. $1444y_{s+3}-102524y_{s+1}=606480x_{s+1}$
25. $38y_{s+3}-38y_{s+1}=2660x_{s+2}$
26. $1444y_{s+3}x_{s+1}-102524y_{s+1}x_{s+1}=606480x_{s+1}^2$
27. $38y_{s+3}x_{s+2}-38y_{s+1}x_{s+2}=2660x_{s+2}^2$
28. $8664y_{s+3}-102524y_{s+2}=50540x_{s+1}$
29. $1444y_{s+3}x_{s+1}-102524y_{s+2}x_{s+1}=50540x_{s+1}^2$
30. $8664y_{s+3}x_{s+1}-102524y_{s+2}x_{s+1}=50540x_{s+1}^2$
31. $38y_{s+3}x_{s+2}-228y_{s+2}x_{s+2}=1330x_{s+2}^2$

REMARKABLE OBSERVATIONS

1. Let N be any non-zero positive integer such that

$$N=\frac{y_{2n+1}-1}{2}$$

It is seen that $8t_{3,N}+1=y_{2n+1}^2$

Similarly $8t_{3,M}+1=x_{2n}^2$

2. Let m and n be two non-zero distinct positive integer such that

$m=x_n+2y_n, n=x_n$

Note that m>n>0 treat m, n as the generators of the Pythagorean triangle T(α,β,γ) .

Where α=2mn, $\beta=m^2-n^2$ and $\gamma=m^2+n^2$

Let A, P represent the area and perimeter of T(α,β,γ). Then the following interesting relations are observed:

(1). $\alpha-70\beta+69\gamma+4=0$

(2). $71\beta-70\gamma-4\frac{A}{P}=4$

(3). $\gamma-71\alpha+280\frac{A}{P}=4$

Each of the following represents the hyperbola:

S. No	Hyperbola	$f_s(=x), g_s(=y)$
1	$x^2-140y^2=1444$	$\frac{1}{19}[70x_{s+1}-8y_{s+1}]$, $\frac{2\sqrt{35}}{19}[y_{s+1}-4x_{s+1}]$
2	$x^2-35y^2=1444$	$\frac{1}{19}[118x_{s+1}-8x_{s+2}]$, $\frac{\sqrt{35}}{19}[2x_{s+2}-20x_{s+1}]$
3	$4x^2-35y^2=51984$	$\frac{1}{57}[352x_{s+1}-2x_{s+3}]$, $\frac{\sqrt{35}}{144}[x_{s+3}-119x_{s+1}]$
4	$35x^2-35y^2=1444$	$\frac{\sqrt{35}}{19}[2\sqrt{35}x_{s+1}-\frac{8}{\sqrt{35}}y_{s+1}]$, $\frac{\sqrt{35}}{19}[2y_{s+1}-8x_{s+1}]$
5	$35x^2-35y^2=12996$	$\frac{\sqrt{35}}{57}[10\sqrt{35}x_{s+1}-\frac{2}{\sqrt{35}}y_{s+2}]$, $\frac{\sqrt{35}}{57}[y_{s+2}-59x_{s+1}]$
6	$35x^2-35y^2=7279204$	$\frac{\sqrt{35}}{1349}[238\sqrt{35}x_{s+1}-\frac{8}{\sqrt{35}}y_{s+3}]$, $\frac{\sqrt{35}}{1349}[2y_{s+3}-1408x_{s+1}]$
7	$x^2-35y\ \ =1444$	$\frac{1}{19}[1408x_{s+2}-118x_{s+3}]$, $\frac{\sqrt{35}}{19}[20x_{s+3}-238x_{s+2}]$
8	$35x^2-35y^2=12996$	$\frac{\sqrt{35}}{57}[\sqrt{35}x_{s+2}-\frac{59}{\sqrt{35}}y_{s+1}]$, $\frac{\sqrt{35}}{57}[10y_{s+1}-4x_{s+2}]$
9	$35x^2-35y^2=1444$	$\frac{\sqrt{35}}{19}[20\sqrt{35}x_{s+2}-\frac{118}{\sqrt{35}}y_{s+2}]$, $\frac{\sqrt{35}}{19}[20y_{s+2}-118x_{s+2}]$
10	$35x^2-35y^2=12996$	$\frac{\sqrt{35}}{57}[119\sqrt{35}x_{s+2}-\frac{59}{\sqrt{35}}y_{s+3}]$ $\frac{\sqrt{35}}{57}[10y_{s+3}-704x_{s+2}]$
11	$35x^2-35y^2=7279204$	$\frac{\sqrt{35}}{1349}[2\sqrt{35}x_{s+3}-\frac{1408}{\sqrt{35}}y_{s+1}]$, $\frac{\sqrt{35}}{1349}[238y_{s+1}-8x_{s+3}]$
12	$35x^2-35y^2=12996$	$\frac{\sqrt{35}}{57}[10\sqrt{35}x_{s+3}-\frac{704}{\sqrt{35}}y_{s+2}]$, $\frac{\sqrt{35}}{57}[119y_{s+2}-59x_{s+3}]$
13	$35x^2-35y^2=1444$	$\frac{\sqrt{35}}{19}[238\sqrt{35}x_{s+3}-\frac{1408}{\sqrt{35}}y_{s+3}]$, $\frac{\sqrt{35}}{19}[238y_{s+3}-1408x_{s+3}]$
14	$35x^2-35y^2=50540$	$\frac{1}{19}[2y_{s+2}-20y_{s+1}]$, $\frac{1}{19\sqrt{35}}[118y_{s+1}-8y_{s+2}]$
15	$35x^2-y^2=1819440$	$\frac{1}{114}[y_{s+3}-119y_{s+1}]$, $\frac{1}{19\sqrt{35}}[1408y_{s+2}-118y_{s+3}]$
16	$35x^2-35y^2=50540$	$\frac{1}{19}[20y_{s+3}-238y_{s+2}]$, $\frac{1}{19\sqrt{35}}[1408y_{s+2}-118y_{s+3}]$

Each of the following represents the parabola

S. No	Parabola	$f_s^2 (= x), g_s (= y)$
1	$140y^2 = 19x - 1444$	$\frac{1}{19}[70x_{2s+2} - 8y_{2s+2}] + 2, \frac{2\sqrt{35}}{19}[y_{s+1} - 4x_{s+1}]$
2	$35y^2 = 19x - 1444$	$\frac{1}{19}[118x_{2s+2} - 8x_{2s+3}] + 2, \frac{\sqrt{35}}{19}[2x_{s+2} - 20x_{s+1}]$
3	$35y^2 = 228x - 51984$	$\frac{1}{57}[352x_{2s+2} - 2x_{2s+4}] + 2, \frac{\sqrt{35}}{144}[x_{s+3} - 119x_{s+1}]$
4	$35y^2 = 19x - 1444$	$\frac{\sqrt{35}}{19}[2\sqrt{35}x_{2s+2} - \frac{8}{\sqrt{35}}y_{2s+2}] + 2, \frac{\sqrt{35}}{19}[2y_{s+1} - 8x_{s+1}]$
5	$35y^2 = 57x - 12996$	$\frac{\sqrt{35}}{57}[10\sqrt{35}x_{2s+2} - \frac{2}{\sqrt{35}}y_{2s+3}] + 2, \frac{\sqrt{35}}{57}[y_{s+2} - 59x_{s+1}]$
6	$35y^2 = 1349x - 7279204$	$\frac{\sqrt{35}}{1349}[238\sqrt{35}x_{2s+2} - \frac{8}{\sqrt{35}}y_{2s+4}] + 2, \frac{\sqrt{35}}{1349}[2y_{s+3} - 1408x_{s+1}]$
7	$35y \;\; = 19x - 1444$	$\frac{1}{19}[1408x_{2s+3} - 118x_{2s+4}] + 2, \frac{\sqrt{35}}{19}[20x_{s+3} - 238x_{s+2}]$
8	$35y^2 = 57x - 12996$	$\frac{\sqrt{35}}{57}[\sqrt{35}x_{2s+3} - \frac{59}{\sqrt{35}}y_{2s+2}] + 2, \frac{\sqrt{35}}{57}[10y_{s+1} - 4x_{s+2}]$
9	$35y^2 = 19x - 1444$	$\frac{\sqrt{35}}{19}[20\sqrt{35}x_{2s+3} - \frac{118}{\sqrt{35}}y_{2s+3}] + 2, \frac{\sqrt{35}}{19}[20y_{s+2} - 118x_{s+2}]$
10	$35y^2 = 57x - 12996$	$\frac{\sqrt{35}}{57}[119\sqrt{35}x_{2s+3} - \frac{59}{\sqrt{35}}y_{2s+4}] + 2, \frac{\sqrt{35}}{57}[10y_{s+3} - 704x_{s+2}]$
11	$35y^2 = 1349x - 7279204$	$\frac{\sqrt{35}}{1349}[2\sqrt{35}x_{2s+4} - \frac{1408}{\sqrt{35}}y_{2s+2}] + 2, \frac{\sqrt{35}}{1349}[238y_{s+1} - 8x_{s+3}]$
12	$35y^2 = 57x - 12996$	$\frac{\sqrt{35}}{57}[10\sqrt{35}x_{2s+4} - \frac{704}{\sqrt{35}}y_{2s+3}] + 2, \frac{\sqrt{35}}{57}[119y_{s+2} - 59x_{s+3}]$
13	$35y^2 = 19x - 1444$	$\frac{\sqrt{35}}{19}[238\sqrt{35}x_{2s+4} - \frac{1408}{\sqrt{35}}y_{2s+4}] + 2, \frac{\sqrt{35}}{19}[238y_{s+3} - 1408x_{s+3}]$
14	$y^2 = 665x - 50540$	$\frac{1}{19}[2y_{2s+3} - 20y_{2s+2}] + 2, \frac{1}{19\sqrt{35}}[118y_{s+1} - 8y_{s+2}]$
15	$y^2 = 3990x - 1819440$	$\frac{1}{114}[y_{2s+4} - 119y_{2s+2}] + 2, \frac{1}{19\sqrt{35}}[1408y_{s+2} - 118y_{s+3}]$
16	$y^2 = 665x - 50540$	$\frac{1}{19}[20y_{2s+4} - 238y_{2s+3}] + 2, \frac{1}{19\sqrt{35}}[1408y_{s+2} - 118y_{s+3}]$

CONCLUSION

In this paper, We have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2 = 35x^2 - 19$. As the binary quadratic Diophantine equation are rich in variety , one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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