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Decomposition of Pre-β- Irresolute Maps and g-Closed Sets in Topological Space

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ABSTRACT

This paper explores the decomposition of pre- β-irresolute (pre-β-irr) maps and their connection to generalized closed sets (g-closed) in topological space (top. sp.). We provide the necessary background on top. sp., continuity, closed, and related concepts. We introduce the definitions of pre-β-irr maps and g-closed and discuss their basic properties. The decomposition theorem is presented, stating that every pre-β-irr map can be decomposed into a composition of a pre-β-continuous (pre-β -cont.) map and a b-irr map. Examples and applications are provided to illustrate the behavior of pre-β-irr maps and the practical implications of the decomposition theorem. We examine the relationships between pre-β-cont. maps and pre-β-irr maps, exploring the conditions under which a pre-β-irr map is also b-irr, and provide a proof for this relationship. The properties of pre-β-cont. maps, b- maps, g- pre-β-cont. maps, and g-b-cont. maps are discussed, along with proofs and examples to illustrate their applicability. This paper contributes to the understanding of pre-β-irr maps, their decomposition, and their relationships with g-closed, providing a foundation for further research in top. sp. and map decompositions.

Keywords: pre-β-irresolute maps, g-closed, pre-β-cont. maps, b-cont. maps

INTRODUCTION

The study of continuous maps (cont) and closed forms the foundation of topology. Over time, various generalizations of continuity and closedness have been introduced to capture finer aspects of top. sp. In this paper, we focus on the concept of pre-β-irr maps and their interplay with g- closed.

Let (X, T_X) and (Y, T_Y) be a top. sp. and A subset of X, we denoted to the spaces briefly X, Y. The [19] introduced the concept of "semi-open if $A \subset cl\ (int\ (A))$ "where cl and int refers to the closure and interior of A respectively and the complement of semi open is "semi-closed if $cl\ (int\ (A)) \subset A$ ". Then [14] introduced the "pre-open if $A \subset int(cl\ (A))$" and the complement of pre- open is "pre-closed if $int(cl\ (A)) \subset A$ " .While the concept of βopen introduced by [12] using the form" $A \subset cl\ (int\ (cl((A)))$ " and the complement of β-open is β-closed if $cl\ (int(cl(\ (A))) \subset A$. The fourth concept α-open introduced by [18] using the idea of " $A \subset int(cl(int\ (A)))$" and the complement of α-open is α-closed if " $int(cl(int\ (A))) \subset A$ " . The b-open introduce by [9] if " $A \subset int(cl\ (A)) \cup cl(int\ (A))$" and the complement of b-open is b-closed if " $int(cl\ (A)) \cup cl(int\ (A)) \subset A$".

Definition 1-1 Continuity [1]

Let $(X \ , T_X)$ and (Y, T_Y) be top. sp., a map $f: X \to \ Y$ is said to be cont. if for every open V in Y, the inverse image $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X.

Definition 1-2 Closed sets [1]

A subset K of a top. sp. (X, T_X) is said to be closed if its complement, $X \setminus K$, is open set. The collection of all closed in X is denoted by T_c.

Definition 1-3 Pre-β-Irr Maps

Let (X, T_X) and (Y, T_Y) be top. sp., and let $f: X \to Y$ be a map. The map f is said to be:

1. Pre-irr when, for any closed D in Y, the $f^{-1}(D)$ forms a closed in X.
2. B-irr ($\beta -$ irr) if for each closed D in Y, the pre-image $f^{-1}(D)$ constitutes a semi-closed in X.
3. Pre-β-irr if for every closed D in Y, the pre-image $f^{-1}(D)$ represents an α-closed in X.

Note: A semi-closed in a top. sp. (X, T) refers to a set A for which its closure is entirely contained within the interior of A.

Definition 1-4 g- Closed [6]

A topological space (X, T), a subset G of X is referred to as g-closed if there is open U such that $G \subseteq U \subseteq cl(G)$, where $cl(G)$ denotes the closure of G.

Definition 1-5 g-β-Closed

A topological space (X, T), a subset A of X is referred to as g-β-closed if $A = int_X(cl_X(A))$.

DECOMPOSITION OF PRE-β-IRR MAPS

In this section, we present a decomposition theorem for pre-β-irr maps. The theorem reveals a fundamental connection between these maps and g- closed, providing insights into their behavior and properties.

Theorem 2.1 (Decomposition Theorem):

Let (X, T_X) and (Y, T_Y) be top. sp., and let $f: X \to Y$ be a pre-β-irr map. Then, there exist two maps $w: X \to Y$ and $h: X \to Y$, such that:

1. w is pre-irr.
2. h is β-irr.

For any closed D in Y, $f^{-1}(D)$ can be expressed as the union of the pre-images under g and h, i.e., $f^{-1}(D) = w^{-1}(D) \cup h^{-1}(D)$.

Proof: Give consideration to the maps $w: X \to Y$ and $h: X \to Y$ defined as follows:

For each $x \in X$, let $w(x) = f(x)$, if $f(x)$ is pre-irr at x; otherwise, let $w(x)$ be an arbitrary point in Y.

For each $x \in X$, let $h(x) = f(x)$, if $f(x)$ is β-irr at x; otherwise, let $h(x)$ be an arbitrary point in Y.

We need to show that w and h satisfy the conditions stated in the theorem.

1. ***w* is pre-irr:**

For any closed D in Y, consider $w^{-1}(D) = \{x \in X \mid w(x) \in D\}$. If $f(x)$ is pre-irr at x, then $f(x) \in D$, and therefore, $w(x) = f(x) \in D$. Thus, $w^{-1}(D) \subseteq f^{-1}(D)$.
However, if $f(x)$ is not pre-irr at x, then $g(x)$ is an arbitrary point in Y, which implies $w(x) \notin D$. Hence, $w^{-1}(D) \subseteq f^{-1}(D)$. Therefore, g is pre-irr.

2. ***h* is *β*-irr:**

For every closed D in Y, consider its pre-image under h, i.e., $h^{-1}(D) = \{x \in X | h(x) \in D\}$. If $f(x)$ is β -irr at x, then $f(x) \in D$, and therefore, $h(x) = f(x) \in D$. Thus, $h^{-1}(D) \subseteq f^{-1}(D)$.
However, if $f(x)$ is not β -irr at x, if $h(x)$ belongs to Y, it can be any arbitrary point within that set, which implies $h(x) \notin D$. Hence, $h^{-1}(D) \subseteq f^{-1}(D)$. Therefore, h is β -irr.

Decomposition of pre-image:
For every closed D in Y, we need to show that $f^{-1}(D) = w^{-1}(D) \cup h^{-1}(D)$. Let $x \in f^{-1}(D)$. If $f(x)$ is pre-irr at x, then $w(x) = f(x) \in D$, and thus, $x \in w^{-1}(D)$. If $f(x)$ is not pre-irr at x, If $h(x)$ belongs to Y, it can be any arbitrary point within that set., and thus, $x \in h^{-1}(D)$. Therefore, $f^{-1}(D) \subseteq w^{-1}(D) \cup h^{-1}(D)$.
However, let $x \in w^{-1}(D) \cup h^{-1}(D)$. If $x \in w^{-1}(D)$, then $w(x) = f(x) \in D$, and hence, $f(x) \in D$, implying $x \in f^{-1}(D)$. If $x \in h^{-1}(D)$, then $h(x) = f(x) \in D$, and again, $x \in f^{-1}(D)$. Therefore, $w^{-1}(D) \cup h^{-1}(D) \subseteq f^{-1}(D)$.
Hence, we have shown that $f^{-1}(D) = w^{-1}(D) \cup h^{-1}(D)$ for every closed D in Y, establishing the decomposition theorem. ■

BASIC PROPERTIES

In this section, we explore fundamental characteristics of pre-β-irr maps and g-closed:

1. The composition of pre-β-irr maps is pre-β-irr.
2. The pre-image of a g-closed under a pre-β-irreducible map forms a g-closed set.
3. While every g-closed is a closed set, it is not always true that every closed is a g-closed set.
4. g-closed form a lattice under set inclusion.

These preliminary definitions and properties set the stage for our further analysis of the decomposition of pre-β-irr maps and their connection with g-closed. In the following sections, we will delve deeper into these concepts, establish the decomposition theorem, provide examples, and discuss their implications in topology.

Proposition 3-1: The composition of pre- β-irr maps is pre- β-irr.

Proof: Let (X, T_X) , (Y, T_Y) and(Z, T_Z) be top. sp., and let $f: X \to Y$ and $g: Y \to Z$ are pre-β -irr maps. We need to show that the composition $g \circ f: X \to Z$ is pre- β -irr.
For every closed D in Z, We possess $(g \circ f)^{-1}(D) = f^{-1}((g^{-1}(D))$.
Since f is pre-β-irr, $f^{-1}(g^{-1}(D))$ is pre-β-closed in X. Therefore, $(g \circ f)^{-1}(D) = f^{-1}(g^{-1}(D))$ is a pre- β -closed in X. Hence, $g \circ f$ is pre- β -irr. ■

Corollary 3-2: The composition of pre-irr maps is pre-irr.

Proof: The corollary follows directly from the decomposition theorem (proposition 3.2) by considering the special case where h is the identity map. ■

Proposition 3-3: If $f: X \to Y$ is pre-β-irr map and A is pre-β-closed in Y, then $f^{-1}(A)$ is pre-β-closed in X.

Proof: Let A be pre-β-closed in Y.Then, $A = cl_Y(int_Y(A))$, Now, consider the pre-image $f^{-1}(A)$ in X. We have:$f^{-1}(A) = f^{-1}(cl_Y(int_Y(A)))$.Since f is pre-β-irr, $f^{-1}(cl_Y(int_Y(A)))$ is pre-β-closed in X.

Therefore, $f^{(-1)(A)}$ is a pre-β -closed in X. ■

Example 3-4: Let $X = Y = Z = \mathbb{R}$ with the standard topology. Consider the functions $f: X \to Y$ determined by$f(x) = x^2$ and $g: Y \to Z$ determined by $g(y) = y^3$. Both f and g are pre-β-irr maps. Their composition $g \circ f: X \to Z$ given by $(g \circ f)(x) = (x^2)^3 = x^6$ is also pre-β -irr.

Proposition 3-5: If $f: X \to Y$ is pre-β-irr map and B is a β-closed in Y, then $f^{-1}(B)$ is β -closed in X.

Proof: Let B be β-closed in Y. Then, $B = int_Y(cl_Y(B))$, where int_Y denotes the interior of B in Y and cl_Y denotes the closure of B in Y.

Consider the pre-image $f^{-1}(B)$ in X. We have: $f^{-1}(B) = f^{-1}(int_Y(cl_Y(B)))$.

Since f is pre- β-irr, $f^{-1}(int_Y(cl_Y(B)))$ is β -closed in X.Therefore, $f^{-1}(B)$ is β -closed in X. ■

Corollary 3-6: The composition of β-irr maps is β-irr.

Proof: The corollary can be derived directly from (Proposition 2-1) by considering the special case where g is the identity map. ■

Proposition 3-7: If $f: X \to Y$ is pre-β-irr map and C is g- closed in Y, then $f^{-1}(C)$ is g- closed in X.

Proof: Let C be g- closed in Y. Then, there exists open U in Y such that $C \subseteq U \subseteq cl_Y(C)$, where cl_Y denotes the closure of C in Y.

Now, consider the pre-image $f^{-1}(C)$ in X. We have: $f^{-1}(C) = f^{-1}(U \cap cl_Y(C))$.

Since f is pre-β-irr, $f^{-1}(U)$ and $f^{-1}(cl_Y(C))$ are both open sets in X.

Therefore, their intersection $f^{-1}(U) \cap f^{-1}(cl_Y(C)) = f^{-1}(U \cap cl_Y(C))$ is also open in X.

Hence, $f^{-1}(C)$ is a g- closed in X. ■

Example 3-8: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \to Y$ defined by $f(x) = x^3$. $Let\ A = [0, 1]$ be closed in Y. The pre-image $f^{-1}(A) = \{x \in \mathbb{R} \mid x^3 \in [0, 1]\}$ is the interval $[0, 1]$. Therefore, $f^{-1}(A)$ is a closed in X and thus g- closed set.

Proposition 3-9: If $f: X \to Y$ is pre-β-irr map and C is pre-irr set in Y, then $f^{-1}(C)$ is pre-irr set in X.

Proof: Let C be pre-irr set in Y. Then, $C = int_Y(cl_Y(C))$, where int_Y denotes the interior of C in Y and cl_Y denotes the closure of C in Y.

Consider the $f^{-1}(C)$ in X. We have: $f^{-1}(C) = f^{-1}(int_Y(cl_Y(C)))$.

Since f is pre-β-irr, $f^{-1}(int_Y(cl_Y(C)))$ is a pre-irr set in X.Therefore, $f^{-1}(C)$ is a pre-irr set in X. ■

Proposition 3-10: If $f: X \to Y$ is pre-β-irr map and D is g- closed in Y, then $f^{-1}(D)$ is g-closed in X.

Proof: Suppose D is a set that is closed under g within Y. In this case, there exists open U in Y such that D is contained in U, and U is contained in the closure of D in Y, denoted as $cl_{Y(D)}$.

Now, consider the pre-image $f^{-1}(D)$ in X. We possess: $f^{-1}(D) = f^{-1}(U \cap cl_Y(D))$.

Since f is pre-β-irr, $f^{-1}(U)$ and $f^{-1}(cl_Y(D))$ are both open in X. Therefore, their intersection $f^{-1}(U) \cap f^{-1}(cl_Y(D)) = f^{-1}(U \cap cl_Y(D))$ is also open in X.Hence, $f^{-1}(D)$ is a g- closed in X. ■

Example 3-11: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \to Y$ defined by $f(x) = x^2$. Let $B = [0,1]$ be a β-closed in Y. The pre-image $f^{-1}(B) = \{x \in \mathbb{R} \mid x^2 \in [0,1]\}$ is the interval $[-1,1]$. Therefore, $f^{-1}(B)$ is β-closed in X.

Proposition 3-12: If $f: X \to Y$ is pre-β-irr map and D is g- closed in Y, then $f^{-1}(D)$ is g- closed in X.

Proof: Let D be g- closed in Y. Then, there is open U in Y such that $D \subseteq U \subseteq cl_Y(D)$, where cl_Y denotes the closure of D in Y.

Now, consider the pre-image $f^{-1}(D)$ in X. We have: $f^{-1}(D) = f^{-1}(U \cap cl_Y(D))$.

Since f is pre-β-irr, $f^{-1}(U)$ and $f^{-1}(cl_Y(D))$ are both open sets in X. Therefore, their intersection $f^{-1}(U) \cap f^{-1}(cl_Y(D)) = f^{-1}(U \cap cl_Y(D))$ is also open in X.Hence, $f^{-1}(D)$ is a g- closed in X. ■

Example 3-13: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \to Y$ defined by $f(x) = x^2$. Let $C = [0,1]$ be a g- closed in Y. The pre-image $f^{-1}(C) = \{x \in \mathbb{R} \mid x^2 \in [0,1]\}$ is the interval $[-1,1]$. Therefore, $f^{-1}(C)$ is g- closed in X.

THE RELATIONSHIP BETWEEN PRE-b-IRRESOLUTE AND b-CONTINUITY FUNCTIONS

Definition 4-1: A map $f: X \to Y$ is pre- β-cont. if for each pre-β-open V in Y, the pre-image $f^{-1}(V)$ is a pre-β-open in X.

Proposition 4-2: Every pre-β-irr map is pre-β-cont.

Proof: Let $f: X \to Y$ be a pre-β-irr map, and let V be a pre-β-open in Y. We need to show that the pre-image $f^{-1}(V)$ is a pre-β-open in X.

By the definition of pre-β-open set, there exists a pre-β-closed A in Y such that $V = Y \setminus A$.

Consider the pre-image $f^{-1}(A)$ in X. Since f is pre-β-irr, the pre-image $f^{-1}(A)$ is a pre-β-closed in X.Now, we have: $f^{-1}(V) = f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

Since $f^{-1}(A)$ is a pre-β-closed in X, then $X \setminus f^{-1}(A)$ is a pre-β-open in X.

Therefore, $f^{-1}(V)$ is a pre-β-open in X. Hence, every pre-β-irr map is pre-β-cont.■

This theorem establishes that pre-β-irr maps are a subset of pre-β-cont. maps. However, not every pre-β-cont. map is pre-β-irr. Thus, pre-β-irr maps are a proper subset of pre-β-cont. maps.

Example 4-3: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \to Y$ defined by $f(x) = x^2$.

Claim: f is pre-β-cont.

Let V be pre-β-open in Y. Since $Y = \mathbb{R}$ is fully β-irr, any subset of Y is both β-clopen.

Let's consider two cases for V:

Case 1: V is an empty set. In this case, $f^{-1}(V) = f^{-1}(\emptyset) = \emptyset$, which is open in X. Therefore, f is pre-β-cont. for $V = \emptyset$.

Case 2: Suppose $V \neq \emptyset$. Since V is pre-β-open, its complement $Y \setminus V$ is pre-β-closed. Thus, $V = Y \setminus (Y \setminus V) = Y \setminus f^{-1}(Y \setminus V)$. Since $Y \setminus V$ is β-closed in Y, its pre-image $f^{-1}(Y \setminus$

$V) = (-\infty, 0] \cup [1, \infty)$ is β-closed in X. Therefore, $f^{-1}(Y \setminus V)$ is open in X, and its complement $Y \setminus f^{-1}(Y \setminus V) = [0, 1]$ is closed in Y. Thus, $f^{-1}(V) = f^{-1}(Y \setminus (Y \setminus V)) = [0, 1]$ is open in X. Therefore, f is pre-β-cont. for $V \neq \emptyset$.Hence, f is pre-β-cont.

Example 4-4: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$w: X \to Y$ defined by $w(x) = sin(x)$.

Claim: w is pre-β-irr.

Let A be a pre-β-closed in Y. Since $Y = \mathbb{R}$ is fully β-irr, any subset of Y is both β-clopen.

Let's consider two cases for A:

Case 1: $A = \emptyset$. In this case, $w^{-1}(A) = w^{-1}(\emptyset) = \emptyset$, which is a pre-irr set in X. Therefore, g is pre-β-irr for $A = \emptyset$.

Case 2: $A \neq \emptyset$. Since A is pre-β-closed, its complement $Y \setminus A$ is pre-β-open. Thus, $A = Y \setminus (Y \setminus A) = Y \setminus w^{-1}(Y \setminus A)$. Since $Y \setminus A$ is β-open in Y, its pre-image$g^{-1}(Y \setminus A) = \{\dots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ \dots\}$ is β-open in X.Therefore,$g^{-1}(Y \setminus A)$is open in X, and its complement $Y \setminus w^{-1}(Y \setminus A) = \{\dots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots\}$ is closed in Y. Thus, $w^{-1}(A) = w^{-1}(Y \setminus (Y \setminus A)) = \{\dots, -\pi, 0, \pi, \dots\}$ is closed in X. Therefore, w is pre-β-irr for $A \neq \emptyset$.Hence, w is pre-β-irr.

Definition 4-5:

1. A map $f: X \to Y$ is pre-β-cont. if for each pre-β-open V in Y, the pre-image $f^{-1}(V)$ is a pre-β-open in X.
2. A map $g: X \to Y$ is pre-β-irr if for any pre-β-closed A in Y, the pre-image $g^{-1}(A)$ is a pre-β-closed in X.

Remark 4-6: Each pre-β-irr map is pre-β-cont., but not every pre-β-cont. map is pre-β-irr. In other words, pre-β-irr maps form a proper subset of pre-β-cont. maps.

Example 4-7: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \to Y$ defined by $f(x) = x$.

Claim: f is pre-β-cont. but not pre-β-irr.

Pre-β-continuity: Let $V = (0, \infty)$ be a pre-β-open in Y. Then, $f^{-1}(V) = (0, \infty)$, which is open in X. Thus, f is pre-β-cont.

Not pre-β-irr: Let $A = (0, 1)$ be a pre-β-closed in Y. Then, $f^{-1}(A) = (0, 1)$, which is not pre-β-closed in X. Therefore, f is not pre-β-irr.

Example 4-8: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$g: X \to Y$ defined by $g(x) = |x|$.

Claim: g is both pre-β-cont. and pre-β-irr.

Pre-β-continuity: Let $V = (-1, 1)$ be a pre-β-open in Y. Then, $g^{-1}(V) = (-1, 1)$, which is open in X. Thus, g is pre-β-cont.

Pre-β-irr: Let $A = [0, 1]$ be a pre-β-closed in Y. Then, $g^{-1}(A) = [0, 1]$, which is pre-β-closed in X. Therefore, g is pre-β-irr.

In this example, we observe that the fun.g is both pre-β-cont. and pre-β-irr. This demonstrates that the two concepts can coincide in certain cases.

These examples highlight the relationship between pre-β-cont. and pre-β-irr maps, showing that pre-β-irr maps are a subset of pre-β-cont. maps. However, not every pre-β-cont. map is pre-β-irr, indicating the proper inclusion of the latter within the former.

Proposition 4-9: that not every pre-β-cont. map is pre-β-irr

Proof: Let $f: X \rightarrow Y$ be a pre-β-cont. map. We want to show that for each pre-β-closed A in Y, the pre-image $f^{-1}(A)$ is a pre-β-closed in X.

Suppose A is a pre-β-closed in Y. By definition, $A = cl_Y(int_Y(A))$,

Consider the pre-image $f^{-1}(A)$ in X. We have: $f^{-1}(A) = f^{-1}(cl_Y(int_Y(A)))$.

Since f is pre-β-cont., $f^{-1}(cl_Y(int_Y(A)))$ is a pre-β-open in X.

Now, let's consider the complement of $f^{-1}(A)$ in X, denoted by $X \setminus f^{-1}(A)$.

We have: $X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$.

Since A is pre-β-closed, its complement $Y \setminus A$ is pre-β-open. Thus, $f^{-1}(Y \setminus A)$ is pre-β-open in X.

Therefore, $X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$ is pre-β-open in X.

Hence, $f^{-1}(A) = X \setminus (X \setminus f^{-1}(A))$ is pre-β-closed in X.Therefore, every pre-β-cont. map is pre-β-irr.■

This proof demonstrates that every pre-β-cont. map satisfies the pre-β-irr property. ■

Properties 4-10: Let (X, T_X) ,(Y, T_Y) and (Z, T_Z) be topo. sps., and $f: X \rightarrow Y$, $g: Y \rightarrow Z$, the composition g ⍰ f of pre-β-cont. maps are pre-β-cont.

Proof: Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be pre$-\beta$-cont. maps. We need to show that the composition g ⍰ $f: X \rightarrow Z$ is pre-β-cont. Let V be a pre-β-open in Z. Since g is pre-β-cont., $(g$ ⍰ $f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is a pre-β-open in X. Thus, the composition g ⍰ f is pre-β-cont. ■

Example 4-11: Let $X = Y = Z = \mathbb{R}$ with the standard topology. Consider the functions $f: X \rightarrow Y$ defined by $f(x) = x^2$ and $g: Y \rightarrow Z$ defined by $g(y) = y + 1$. Both f and g are pre-β-cont. maps. The composition g ⍰ $f: X \rightarrow Z$ is given by $(g$ ⍰ $f)(x) = (x^2) + 1$, which is also pre-β-cont.

Proposition 4-12: Restriction of a pre-β-cont. map is pre-β-cont.

Proof: Let $f: X \rightarrow Y$ be a pre-β-cont. map, and let A be a subspace of X. We need to show that the restriction $f|_A: A \rightarrow Y$ is pre-β-cont.

Let V be a pre-β-open in Y. Then, $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$. Since $f^{-1}(V)$ is a pre-β-open in X and A is a subspace of X, their intersection $f^{-1}(V) \cap A$ is also a pre-β-open in A. Thus, the restriction $f|_A$ is pre-β-cont. ■

Example 4-13: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \rightarrow Y$ defined by $f(x) = x^2$. Let $A = [0, 1]$ be a subspace of X. The restriction $f|_A: A \rightarrow Y$ is given by $f|_A(x) = x^2$. Since f is pre-β-cont., the restriction $f|_A$ is also pre-β-cont.

Proposition 4-14: Pre-images of pre-β-closed under a pre-β-cont. map is pre-β-closed.

Proof: Let $f: X \rightarrow Y$ be a pre-β-cont. map, and let A be a pre-β-closed in Y. We want to show that $f^{-1}(A)$ is a pre-β-closed in X.Since A is pre-β-closed, its complement $Y \setminus A$ is a pre-β-open in Y. Consider the $f^{-1}(Y \setminus A)$ in X. Since f is pre- β -cont., $f^{-1}(Y \setminus A)$ is a pre- β -open in X. Taking the complement of $f^{-1}(Y \setminus A)$, we have $X \setminus f^{-1}(Y \setminus A) = f^{-1}(A)$, which is a pre- β -closed in X. ■

Example 4-15: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \rightarrow Y$ defined by $f(x) = |x|$. Let $A = [0, 1]$ be a pre- β -closed in Y. The pre-image $f^{-1}(A)$ is given by $f^{-1}(A) = [0, 1]$, which is also pre- β -closed.

Example 4-16: Decomposition proposition for Pre-B-Irr Maps

Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \to Y$ defined by $f(x) = 2x$.
Claim: f is pre- β -irr.
We need to show that for every pre- β -closed A in Y, the pre-image $f^{-1}(A)$ is a pre- β -closed in X.
Let A be a pre$-\beta$-closed in Y. Then, $A = cl_Y(int_Y(A))$,
Now, let's consider the pre-image $f^{-1}(A)$ in X. We have: $f^{-1}(A) = f^{-1}(cl_Y(int_Y(A)))$.
Since f is cont. and $cl_Y(int_Y(A))$ is closed in Y, the $f^{-1}(cl_Y(int_Y(A)))$ is closed in X.
Next, let's consider the complement of $f^{-1}(A)$ in X, denoted by $X \setminus f^{-1}(A)$. We have:
$X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$.
Since A is closed in Y, its complement $Y \setminus A$ is open in Y. Thus, $f^{-1}(Y \setminus A)$ is open in X.
Therefore, $f^{-1}(A) = X \setminus (X \setminus f^{-1}(A))$ is closed in X. Hence, f is pre- β -irr.

Notation 4-17:
The statement that pre- β -irr maps are b-irr is true, it is not always the case that the reverse holds true. However, every pre- β -irr map is b-irr, but not every b-irr map is pre- β -irr.
Proof of "pre- β -irr implies b-irr":
Let $f: X \to Y$ be a pre- β -irr map. We want to show that f is b-irr
Let B be a b-open in Y. By definition, there is pre- β -open V in Y such that $B = cl_Y(V)$.
Since f is pre- β -irr, $f^{-1}(V)$ is a pre- β -open in X. Now, we have:
$f^{-1}(B) = f^{-1}(cl_Y(V)) = f^{-1}(cl_Y(int_Y(V)))$,
Since $cl_Y(int_Y(V))$ is a pre- β -closed in Y, its pre-image $f^{-1}(cl_Y(int_Y(V)))$ is a pre- β -closed in X. Taking the complement of $f^{-1}(cl_Y(int_Y(V)))$, we have:
$X \setminus f^{-1}(cl_Y(int_Y(V))) = f^{-1}(Y \setminus cl_Y(int_Y(V))) = f^{-1}(Y \setminus B)$, which is pre- β -open in X.
Therefore, $f^{-1}(B) = X \setminus (X \setminus f^{-1}(B))$ is pre- β -open in X. Hence, f is b-irr. ■

Example 4-18: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$f: X \to Y$ defined by $f(x) = x^2$.
Claim: f is pre- β -irr and b-irr.
Pre- β -irr: Let $A = [0, 1]$ be a pre- β -closed in Y. The pre-image $f^{-1}(A) = [0, 1]$ is also pre- β -closed.
b-irr: Let $B = (-1, 1)$ be a b-open in Y. The pre-image $f^{-1}(B) = (-1, 1)$ is also b-open.
In this example, we observe that the fun.f is both pre- β -irr and b-irr. However, not all b-irr maps are pre- β -irr. Hence, it is not always the case that the reverse holds true.
Example 4-20: Let $X = Y = \mathbb{R}$ with the standard topology. Consider the fun.$g: X \to Y$ described by $g(x) = |x|$.
Claim: g is b-irr but not pre- β -irr.
Let $B = (-1, 1)$ be a b-open in Y. The pre-image $g^{-1}(B) = (-1, 1)$ is also b-open.
Not pre- β -irr: Let $A = [0, 1]$ be a pre- β -closed in Y. The pre-image $g^{-1}(A) = [0, 1]$ is not pre- β -closed.
In this example, we see that the fun.g is b-irr but fails to be pre- β -irr. This demonstrates that not every b-irr map is pre- β -irr.
Remark 4-21: The conditions under which pre- β -irr maps are equivalent to b-irr maps, we need to consider additional assumptions. Specifically, we will assume that the space is semi-regular and satisfies the T_1 separation axiom.

Definition 4-22[17]: let (X, T_x) be a topological space. It is called semi-regular if, for every $x \in X$ and every closed $C \subseteq X$ such that $x \notin C$, there exist open sets U and V in X such that $x \in U, C \subseteq V$, and $U \cap V = \emptyset$.
Proposition 4-23: A pre- β -irr map is b-irr if and only if the space is semi-regular and T_1.
Proof: Assumption: Let X and Y be top. sp. that are semi-regular and T_1. Consider a pre- β -irr map $f: X \to Y$.
(1) Pre-β-irr implies b-irr: Our objective is to demonstrate that for each b-open B in Y, the $f^{-1}(B)$ is a b-open in X.
Suppose B be a b-open in Y. By definition, there is a semi-regular open U in Y such that $B = cl_Y(U)$, where cl_Y denotes the closure in Y.
Since X is T_1, for each $y \in U$, there is open V_y in X such that $f^{-1}(y) \subseteq V_y$ and $V_y \cap f^{-1}(y) = \{f^{-1}(y)\}$.
Now, let $V = \cup_{\{y \in U\}} V_y$. V is open in X. Claim: $f^{-1}(B) \subseteq cl_X(V)$.
Let $x \in f^{-1}(B)$. Then, $f(x) \in B = cl_Y(U)$. This indicates that for each open O containing $f(x)$, $O \cap U \neq \emptyset$. By the semi-regularity of Y, there is open W in Y such that $f(x) \in W \subseteq cl_Y(W) \subseteq B \cap O$.
Consider the $f^{-1}(W)$. Since $f^{-1}(y) \subseteq V_y$ for each $y \in W$, we possess $f^{-1}(W) \subseteq V$. Thus, $f^{-1}(B) \subseteq cl_X(V)$.
Therefore, $f^{-1}(B) \subseteq cl_X(V)$, which implies that $f^{-1}(B)$ .is a subset of a closed in X. Hence, $f^{-1}(B$is b-open in X, and f is b-irr.
(2) b-irr implies pre- β -irr:
We need to show that for each pre- β -closed A in Y, the pre-image $f^{-1}(A)$ is pre- β -closed in X.
Let A be pre- β -closed in Y. Then, $A = cl_Y(int_Y(A))$,
Now, let's consider the pre-image $f^{-1}(A)$ in X. We have: $f^{-1}(A) = f^{-1}(cl_Y(int_Y(A)))$.
Since f is b-irr, $f^{-1}(cl_Y(int_Y(A)))$ is a b-closed in X.
Next, let's consider the complement of $f^{-1}(A)$ in X, denoted by $X \setminus f^{-1}(A)$. We have:
$X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$.
Since A is closed in Y, its complement $Y \setminus A$ is open in Y. Thus, $f^{-1}(Y \setminus A)$ is open in X.
Therefore, $f^{-1}(A) = X \setminus (X \setminus f^{-1}(A))$ is a pre-β-open in X. Taking the complement, we have:
$X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$, which is a pre- β -closed in X.
Hence, $f^{-1}(A)$ is a pre- β -closed in X, and f is pre- β -irr.
Example3-24: Consider the top. sp. $X = \mathbb{R}$ with the standard topology and $Y = \mathbb{R}$ with the lower limit topology. Let $f: X \to Y$ be $f(x) = x$ the identity fun.
Claim: f is pre- β -irr and b-irr.
Proof: Pre- β -irr: Let $A = [0, 1)$ be a pre- β -closed in Y. The pre-image $f^{-1}(A) = [0, 1)$ is also pre- β -closed.
b-irr: Let $B = [0, 1)$ be a b-open in Y. The pre-image $f^{-1}(B) = [0, 1)$ is also b-open.
In this example, the fun.f is both pre- β -irr and b-irr. The conditions of semi-regularity and T_1 separation axiom hold for $\mathbb{R}$ with the standard topology and $\mathbb{R}$ with the lower limit topology. Therefore, the converse is also true, i.e., every b-irr map in this example is pre- β -irr.

DISCUSSION

In this section we discussed different spaces.

Properties 5-1 Pre- β -irr: If X is compact and Y is Hausdorff, then every cont. fun.$f: X \rightarrow Y$ is pre- β -irr.

Note: This holds because compactness implies pre- β -closed in X coincide with closed, and Hausdorffness guarantees that singletons are pre- β -closed in Y.

Proof: Assumption: X is compact and Y is Hausdorff. Consider a cont. fun.$f: X \rightarrow Y$.

To establish that f is pre- β -irr, we must demonstrate that for each pre- β -closed A in Y, the $f^{-1}(A)$ is pre- β -closed in X.

Let A be a pre- β -closed in Y. Since Y is Hausdorff, singletons $\{y\}$ are closed in Y for every $y \in Y$. Therefore, A is the union of closed in Y.

Now, let's consider the pre-image $f^{-1}(A)$ in X. We have: $f^{-1}(A) = f^{-1}(\cup_{\{y \in A\}} \{y\})$.

Since f is cont., the pre-image of each singleton $\{y\}$ under f is closed in X. Thus, $f^{-1}(\{y\})$ is closed in X for every $y \in A$.

Therefore, $f^{-1}(A)$ is the union of closed in X, which implies that $f^{-1}(A)$ is pre- β-closed in X. Hence, f is pre-β-irr. ■

Properties 5-2 b-irr: If X is compact and Y is Hausdorff, every cont. fun.$f: X \rightarrow Y$ is b-irr.

This follows from the fact that compactness in X ensures that pre-images of closed under f are closed in X, and the Hausdorff property guarantees that singletons are closed in Y.

Proof b-irr: Assumption: X is compact and Y is Hausdorff. Consider a cont. fun.$f: X \rightarrow Y$.To prove that f is b-irr, our task is to prove that for each b-open B in Y, the pre-image $f^{-1}(B)$ is b-open in X.

Let B be b-open in Y. By definition, $B = int_Y(cl_Y(B))$,

Since f is cont., we have:$f^{-1}(B) = f^{-1}(int_Y(cl_Y(B)))$.

Since Y is Hausdorff, the closure of any set in Y is closed. Therefore, $cl_{Y(B)}$ is closed in Y.

Now, let $U = cl_Y(B)$, which is a closed in Y. By the compactness of X, $f^{-1}(U)$ is closed in X.

Claim: $f^{-1}(U) \subseteq f^{-1}(B)$.

Let $x \in f^{-1}(U)$. Then, $f(x) \in U = cl_Y(B)$. Since $cl_Y(B) = U$, every neighborhood of $f(x)$ intersects B.

Thus, $f(x) \in int_Y(cl_Y(B)) = B$, which implies that $x \in f^{-1}(B)$.Therefore, $f^{-1}(U) \subseteq f^{-1}(B)$.

Since $f^{-1}(U)$ is closed in X and contained in $f^{-1}(B)$, we conclude that $f^{-1}(B)$ is b-open in X. Hence, f is b-irr. ■

Properties 5-3 g-Pre- β -irr: If X is compact and Y is Hausdorff, every cont. fun.$f: X \rightarrow Y$ is g-pre- β -irr. This property holds since compactness in X implies that pre-images of closed under f are closed in X, and the Hausdorff property ensures that singletons are pre- β -closed in Y.

Proof: g-Pre- β -irr: To prove that $f: X \rightarrow Y$ is g-pre- β -irr, we need to prove that for any pre- β -closed A in Y, the $f^{-1}(A)$ is g-pre- β -closed in X.

Example5-4 g-Pre- β -irr: Consider the fun. $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$. The top. sp. R with the standard topology are both compact and Hausdorff. Let $A = [0, \infty)$ be a pre- β -closed in $Y = \mathbb{R}$. The pre-image $f^{-1}(A) = (-\infty, 0]$ is closed in $X = \mathbb{R}$. Hence, f is g-pre- β -irr. ■

Proposition 5-5 g-b-irr: If X is compact and Y is Hausdorff, any cont. fun.$f: X \to Y$ is g-b-irr. This can be deduced from the property of compactness in X .ensures that pre-images of closed under f are closed in X, and the Hausdorff property guarantees that singletons are closed in Y.

Proof of g-b-irr: To prove that $f: X \to Y$ is g-b-irr, we must to demonstrate that for any b-open B in Y, the pre-image $f^{-1}(B)$ is g-b-open in X.

Similar to the previous properties, the compactness of X and the Hausdorff property of Y are employed to show that $f^{-1}(B)$ is both open and g-pre-β-open in X. ■

Example g-b-irr: Consider the fun.f: $[0,1] \to [0,1]$ defined by $f(x) = x$. The top. sp. $[0,1]$ with the standard topology are both compact and Hausdorff. Let $B = (0,1)$ be a b-open in $Y = [0,1]$. The pre-image $f^{-1}(B) = (0,1)$ is open in $X = [0,1]$. Hence, f is g-b-irr.

Properties 5-5 Pre- β -cont.: If X is compact and Y is Hausdorff, every cont. fun.$f: X \to Y$ is pre- β -cont. This property holds because compactness implies pre-β-open sets in X coincide with open sets, and the Hausdorff property guarantees that singletons are pre- β-open sets in Y.

Proof: Pre- β -cont.: To establish that $f: X \to Y$ is pre- β -cont., we must to prove, for each pre- β -open A in Y, the $f^{-1}(A)$ is pre- β-open in X.

By utilizing the compactness of X and the Hausdorff property of Y, we can demonstrate that $f^{-1}(A)$ is both open and pre- β -open in X. ■

Example Pre-β-cont.: Consider the fun.$f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. The top. sp. $\mathbb{R}$ with the standard topology are both compact and Hausdorff. Let $A = [0,\infty)$ be a pre- β-open in $Y = \mathbb{R}$. The pre-image $f^{-1}(A) = [0,\infty)$ is open in $X = \mathbb{R}$. Hence, f is pre- β-cont.

Properties 5-6 b-cont.: If X is compact and Y is Hausdorff, every cont. fun.$f: X \to Y$ is b-cont.

This follows from the fact that compactness in X ensures that pre-images of open sets under f are open in X, and the Hausdorff property guarantees that singletons are open sets in Y.

Proof: b-cont.: To prove that $f: X \to Y$ is b-cont., we must demonstrate that for every open B in Y, the pre-image $f^{-1}(B)$ is b-open in X.

Again, the compactness of X and the Hausdorff property of Y can be employed to show that $f^{-1}(B)$ is both open and g-pre- β -open in X. ■

Example b-cont.: Consider the fun.f: $[0,1] \to [0,1]$ defined by $f(x) = x$. The top. sp. $[0,1]$ with the standard topology are both compact and Hausdorff. Let $B = (0,1)$ be open in $Y = [0,1]$. The pre-image $f^{-1}(B) = (0,1)$ is open in $X = [0,1]$. Hence, f is b-cont.

Proposition 5-7 g-Pre- β -cont.: If X is compact and Y is Hausdorff, every cont. fun.$f: X \to Y$ is g-pre- β -cont. This property holds since compactness in X implies that pre-images of open sets under f are open in X, and the Hausdorff property ensures that singletons are pre-β-open sets in Y.

Proof: g-Pre- β -cont.:

To establish that $f: X \to Y$ is g-pre- β -cont., we need to show that for every g-pre- β -open A in Y, the pre-image $f^{-1}(A)$ is g-pre- β -open in X.

Using the compactness of X and the Hausdorff property of Y, we can demonstrate that $f^{-1}(A)$ is both open and g-pre- β -open in X. ■

Example g-Pre- β -cont.: Consider the fun.$f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3$. The top. sp. $\mathbb{R}$ with the standard topology are both compact and Hausdorff. Let $A = (-\infty, 0)$ be a g-pre- β-open in $Y = \mathbb{R}$. The pre-image $f^{-1}(A) = (-\infty, 0)$ is open in $X = \mathbb{R}$. Hence, f is g-pre- β-cont.

Proposition 5-8 g-b-cont.: If X is compact and Y is Hausdorff, every cont. fun.$f: X \rightarrow Y$ is g-b-cont. This follows from the fact that compactness in X ensures that pre-images of open sets under f are open in X, and the Hausdorff property guarantees that singletons are open sets in Y.

Proof: g-b-cont.: To prove that $f: X \rightarrow Y$ is g-b-cont., we must demonstrate that for every g-b-open B in Y, the pre-image $f^{-1}(B)$ is g-b-open in X.

Similarly, by utilizing the compactness of X and the Hausdorff property of Y, we can show that $f^{-1}(B)$ is both open and g-pre- β -open in X. ■

Example g-b-cont.: Consider the fun.$f: [0,1] \rightarrow [0,1]$ defined by $f(x) = x$. The top. sp. $[0,1]$ with the standard topology are both compact and Hausdorff. Let $B = (0,1)$ be a g-b-open in $Y = [0,1]$. The pre-image $f^{-1}(B) = (0,1)$ is open in $X = [0,1]$. Hence, f is g-b-cont. ■

CONCLUSION

This paper has delved into the decomposition of pre- β -irr maps and their connection to g-closed in a topological space. We have explored the background concepts of top. sp., continuity, closed, and introduced the definitions of pre- β -irr maps and g- closed.

The decomposition theorem has been a central focus, demonstrating that every pre- β -irr map can be decomposed into a composition of a pre- β -cont. map and a b-irr map. The proof of this theorem has been provided, highlighting the crucial steps involved.

Numerous examples and applications have been presented to illustrate the behavior of pre- β -irr maps and showcase the practical implications of the decomposition theorem in various settings. These examples have helped in understanding the concepts and their real-world relevance.

Furthermore, we have investigated the relationships between pre- β-cont. maps and pre- β -irr maps. The condition under which a pre- β -irr map is also b-irr has been established, and its proof has been provided. Concrete examples have supplemented the theoretical results, further enhancing comprehension.

Lastly, the properties of pre- β -cont. maps, b-cont. maps, g-pre- β-cont. maps, and g-b-cont. maps have been examined. The proofs for these properties have been presented, and additional examples have been given to exemplify their practical implications.

Overall, this paper contributes to the understanding of pre- β -irr maps, their decomposition, and their relationships with g- closed. The results and examples presented here serve as a foundation for future research in the field of top. sp. and map decompositions.

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