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Research Paper

Fuzzy Dot Completely Closed BH-Ideal of BH-Algebra

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ABSTRACT

This article introduces the ideas of fuzzy dots in BH-algebras and fuzzy dots ideals in BH-algebras and discusses some of the research's findings.

Keywords: Fuzzy dots, BH-algebras, BCK-algebras

INTRODUCTION

BCK-algebras and BCI-algebras are two types of abstract algebras that Y. Imai and K. Iseki presented [1, 2, and 3]. It is well known that the BCK-algebra class is a proper subclass of the BCI-algebra class .The ideal theory of BCK-algebras is introduced by K. Iseki and S. Tanaka [7]. The concepts of fuzzy relations and fuzzy groups are introduced by P. Bhattacharya, and *et al*. [4]. Y. B. Jun, and et al. introduce the concept of BH-algebras [9]. Since then, BH-algebras have been researched by other writers. Particularly, the fuzzy theory in BH-algebras was studied by Q. Zhang, and et al. [10]. Fuzzy sets and fuzzy groups were introduced by L.A. Zadeh [6] and A. Rosenfeld [8, respectively]. Fuzzy BCK-algebras were first described by O.G. Xi [5]. Following that, Y.B. Jun and J. Meng [10] worked on Characterizing fuzzy subalgebras by their level subalgebras on BCK-algebras. D-algebras were introduced by J. Neggers and H. S. Kim [11] while fuzzy d-algebras were introduced by M. Akram [12].

In this study, the concepts of Fuzzy dot Completely Closed BH-Ideal of BH-Algebra are classified. Then, we look into a number of fundamental aspects of fuzzy BH-ideals and fuzzy dot BH-ideals.

PRELIMINARIES

Definition 2-1: [10]

A BH-algebra is a non-empty set χ with a constant 0 and a binary operation * satisfying the following conditions:

(a) $a * a = 0$, for all $a \in \chi$.

(b) $a * b = 0$ and $b * a = 0$ imply $a = b$, for all $a, b \in \chi$.

(c) $a * 0 = a$, for all $a \in \chi$.

Definition 2-2 [12]:

Let $\mathbb{A} = \{(a, \mathbb{A}(a)) : a \in \chi\}$ and $\mathbb{B} = \{(a, \mathbb{B}(a)) : a \in \chi\}$ be two fuzzy subsets of χ. The Cartesian product $\mathbb{A} \times \mathbb{B} : \chi \times \chi \longrightarrow [0,1]$ is defined by $(\mathbb{A} \times \mathbb{B})(a, b) = \mathbb{A}(a).\mathbb{B}(b)$ for all a, $b \in \chi$.

Definition 2-3 [9]:
A fuzzy subset $\mathbb{A}$ of χ is said to be a fuzzy ideal of χ if satisfies the inequalities:
(a) $\mathbb{A}(0) \geq \mathbb{A}(a)$
(b) $\mathbb{A}(a) \geq \min\{\mathbb{A}(a * b), \mathbb{A}(b)\}$ for all $a, b \in \chi$.

Definition 2-4 [9]:
A fuzzy subset $\mathbb{A}$ of χ is said to be a fuzzy BH-ideal of χ if satisfies the inequalities:
(a) $\mathbb{A}(0) \geq \mathbb{A}(\mathbb{x})$
(b) $\mathbb{A}(a) \geq \min\{\mathbb{A}(a * b), \mathbb{A}(b)\}$ for all $a, b \in \chi$.
(c) $\mathbb{A}(a * b) \geq \min\{\mathbb{A}(a), \mathbb{A}(b)\}$ for all $a, b \in \chi$.

Definition 2-5 [12]:
A fuzzy subset $\mathbb{A}$ of χ is said to be a fuzzy dot subalgebra of χ if $\mathbb{A}(a * b) \geq \mathbb{A}(a).\mathbb{A}(b)$ for all $a, b \in \chi$.

Definition 2-6:
Let χ be a BH-algebra and let $\mathbb{A}$ be a fuzzy subset of χ, then $\mathbb{A}$ is said to be a fuzzy dot BH-ideal if satisfies the following conditions:
(a) $\mathbb{A}(\mathbb{e}) \geq \mathbb{A}(a)$.
(b) $\mathbb{A}(a) \geq \mathbb{A}(a * b).\mathbb{A}(b)$.

Definition 2-7:
Let χ be a BH-algebra and let $\mathbb{A}$ be a fuzzy subset of χ, then $\mathbb{A}$ is said to be fuzzy dot BH-sub-algebra if $\mathbb{A}(a * b) \geq \mathbb{A}(a).\mathbb{A}(b)$.

Example 2-8:
Consider the BH-algebra $\mathbb{X} = \{0,1,2,3\}$ with the following operation table

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Let the fuzzy set $\mathbb{A}$ which is defined as:
$$\mathbb{A} = \begin{cases} 0.6 & a = 0,1 \\ 0.4 & a = 2,3 \end{cases}$$
then $\mathbb{A}$ is:
a fuzzy dot BH-sub-algebra.
a fuzzy dot BH-ideal.

Definition 2-9:
Let χ be a BH-algebra and $\mathbb{A}$ be a fuzzy dot BH-ideal, then $\mathbb{A}$ is said to be a fuzzy dot closed BH-ideal if $\mathbb{A}(0 * a) \geq \mathbb{A}(a)$.

Example 2-10:
The same example (1.8).

Proposition 2-11:
If $\mathbb{A}$ is a fuzzy dot sub-algebra of χ, then $\mathbb{A}(0) \geq \mathbb{A}(a)^{\mathbb{m}}$ for all $a \in \chi$.

Proof. For all $a \in \chi$, we have $a * a = 0$, hence

$$\mathbb{A}(0) = \mathbb{A}(a * a) = \mathbb{A}((a * 0)(a * 0))$$
$$= \mathbb{A}(a * (a * a)) * (a * (a * a))$$
$$= \mathbb{A}(a * 0)\mathbb{A}(a * 0)\mathbb{A}(a * 0) \dots \dots \dots \dots$$

$\mathbb{m}$-time

$$= \mathbb{A}(a)^{\mathbb{m}}$$

Proposition 2-12:
If $\mathbb{A}$ is a fuzzy dot sub-algebra of χ, then $\mathbb{A}^{\mathbb{m}}$ ($\mathbb{m}$ is a positive integer number) is a fuzzy dot sub-algebra.

Proof. For any $a \in \chi$, $\mathbb{A}^{\mathbb{m}}$ is a fuzzy subset of χ defined by $\mathbb{A}^{\mathbb{m}}(a) = \mathbb{A}(a)^{\mathbb{m}}$
Let $\mathbb{A}$ is a fuzzy dot sub-algebra of χ.
$\mathbb{A}(a * b) \geq \mathbb{A}(a).\mathbb{A}(b), \forall\, a, b \in \chi$.
We have
$\mathbb{A}^{\mathbb{m}}(a * b) = \mathbb{A}(a * b)^{\mathbb{m}} \geq (\mathbb{A}(a).\mathbb{A}(b))^{\mathbb{m}} \geq \mathbb{A}(a)^{\mathbb{m}}.\mathbb{A}(b)^{\mathbb{m}} \geq \mathbb{A}^{\mathbb{m}}(a).\mathbb{A}^{\mathbb{m}}(b)$.

Proposition 2-13:
Let χ be an associative BH-algebra, then every fuzzy ideal is a fuzzy dot closed BH-ideal.
Proof. Let $\mathbb{A}$ is a fuzzy ideal and for all $a \in \chi$
$\mathbb{A}(0 * a) = \mathbb{A}(a) \geq \mathbb{A}(a)$ $(0 * a = a \quad \forall\, a \in \chi)$
Then $\mathbb{A}$ is a fuzzy dot closed BH-ideal.

MAIN RESULTS

Definition 3-1:
Let χ be a BH-algebra and $\mathbb{A}$ be a fuzzy dot BH-ideal, then $\mathbb{A}$ is said to be a fuzzy dot completely closed BH-ideal, if $\mathbb{A}(a * b) \geq \mathbb{A}(a).\mathbb{A}(b), \forall a, b \in \chi$.

Example 3-2:
The same example (1.8).

Proposition 3-3:
If $\mathbb{A}$ and $\mathbb{B}$ are fuzzy dot completely closed BH-ideals of a BH-algebra χ, then so is $\mathbb{A} \wedge \mathbb{B}$.

Proof. Let $a, b \in \chi$. Then

$$\mathbb{A} \wedge \mathbb{B}(a * b) = \min\{\mathbb{A}(a * b), \mathbb{B}(a * b)\}$$
$$\geq \min\{\mathbb{A}(a).\mathbb{A}(b), \mathbb{B}(a).\mathbb{B}(b)\}$$
$$\geq (\min\{\mathbb{A}(a), \mathbb{B}(a)\}).(\min\{\mathbb{A}(b), \mathbb{B}(b)\})$$
$$= ((\mathbb{A} \wedge \mathbb{B})(a)).((\mathbb{A} \wedge \mathbb{B})(b))$$

Hence $\mathbb{A} \wedge \mathbb{B}$ is a fuzzy dot completely closed BH-ideal of a BH-algebra χ.

Proposition 3-4:
If $g: \chi \longrightarrow \xi$ is a homomorphism of BH-algebras. If $\mathbb{B}$ is a fuzzy dot completely closed BH-ideal of ξ, then the pre-image $g^{-1}(\mathbb{B})$ of $\mathbb{B}$ under g is a fuzzy dot completely closed BH-ideal of χ.

Proof. Assume that $\mathbb{B}$ is a fuzzy dot completely closed BH-ideal of ξ and let $\mathbb{x}_1, \mathbb{x}_2 \in \chi$, we have,

$$g^{-1}(\mathbb{B})(\mathbb{x}_1 * \mathbb{x}_2) = \mathbb{B}(g(\mathbb{x}_1 * \mathbb{x}_2)) = \mathbb{B}(g(\mathbb{x}_1).g(\mathbb{x}_2))$$

$$\geq \mathbb{B}(g(\mathbb{x}_1)).\mathbb{B}(g(\mathbb{x}_2)) = g^{-1}(\mathbb{B})(\mathbb{x}_1).g^{-1}(\mathbb{B})(\mathbb{x}_2)$$

Then $g^{-1}(\mathbb{B})$ is a fuzzy dot completely closed BH-ideal of a BH-algebra.

Proposition 3-5:

If $g: \chi \longrightarrow \xi$ is a homomorphism of a BH-algebra χ onto a BH-algebra ξ . If $\mathbb{A}$ is a fuzzy dot completely closed BH-ideal of χ, then the image $g(\mathbb{A})$ of $\mathbb{A}$ under g is a fuzzy dot completely closed BH-ideal of ξ.

Proof. Consider that $\mathbb{A}$ is a fuzzy dot completely closed BH-ideal of χ and let $\mathbb{y}_1, \mathbb{y}_2 \in \xi$ and let $\mathbb{u}_1 = g^{-1}(\mathbb{y}_1)$, $\mathbb{u}_2 = g^{-1}(\mathbb{y}_2)$ and $\mathbb{u}_{12} = g^{-1}(\mathbb{y}_1 * \mathbb{y}_2)$. Consider the set

$\mathbb{u}_1 * \mathbb{u}_2 = \{\mathbb{x} \in \chi: \mathbb{x} = \mathbb{a}_1 * \mathbb{a}_2$ for some $\mathbb{a}_1 \in \mathbb{u}_1, \mathbb{a}_2 \in \mathbb{u}_2\}$

If $\mathbb{x} \in \mathbb{u}_1 * \mathbb{u}_2$, then $\mathbb{x} = \mathbb{x}_1 * \mathbb{x}_2$ for some $\mathbb{x}_1 \in \mathbb{u}_1$ and $\mathbb{x}_2 \in \mathbb{u}_2$, so that

$$g(\mathbb{x}) = g(\mathbb{x}_1 * \mathbb{x}_2) = g(\mathbb{x}_1) * g(\mathbb{x}_2) = \mathbb{y}_1 * \mathbb{y}_2$$

that is, $\mathbb{x} \in g^{-1}(\mathbb{y}_1 * \mathbb{y}_2) = \mathbb{u}_{12}$. Hence $\mathbb{u}_1 * \mathbb{u}_2 \subseteq \mathbb{u}_{12}$. It follows that

$$\begin{aligned} g(\mathbb{A})(\mathbb{y}_1 * \mathbb{y}_2) &= \sup\{\mathbb{A}(\mathbb{x}): \mathbb{x} \in g^{-1}(\mathbb{y}_1 * \mathbb{y}_2)\} \\ &= \sup\{\mathbb{A}(\mathbb{x}): \mathbb{x} \in \mathbb{u}_{12}\} \\ &\geq \sup\{\mathbb{A}(\mathbb{x}): \mathbb{x} \in \mathbb{u}_1 * \mathbb{u}_2\} \\ &\geq \sup\{\mathbb{A}(\mathbb{x}_1 * \mathbb{x}_2): \mathbb{x}_1 \in \mathbb{u}_1, \mathbb{x}_2 \in \mathbb{u}_2\} \\ &\geq \sup\{\mathbb{A}(\mathbb{x}_1).\mathbb{A}(\mathbb{x}_2): \mathbb{x}_1 \in \mathbb{u}_1, \mathbb{x}_2 \in \mathbb{u}_2\} \end{aligned}$$

Since $.: [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\widetilde{\mathbb{x}}_1 \geq \sup\{\mathbb{A}(\mathbb{x}_1) - \delta: \mathbb{x}_1 \in \mathbb{u}_1\}$ and $\widetilde{\mathbb{x}}_2 \geq \sup\{\mathbb{A}(\mathbb{x}_2) - \delta: \mathbb{x}_2 \in \mathbb{u}_2\}$, then $\widetilde{\mathbb{x}}_1.\widetilde{\mathbb{x}}_2 \geq \sup\{\mathbb{A}(\mathbb{x}_1): \mathbb{x}_1 \in \mathbb{u}_1\}.\sup\{\mathbb{A}(\mathbb{x}_2): \mathbb{x}_2 \in \mathbb{u}_2\} - \varepsilon$. Choose $\mathbb{a}_1 \in \mathbb{u}_1$ and $\mathbb{a}_2 \in \mathbb{u}_2$ such that $\mathbb{A}(\mathbb{a}_1) \geq \sup\{\mathbb{A}(\mathbb{x}_1) - \delta: \mathbb{x}_1 \in \mathbb{u}_1\}$ and $\mathbb{A}(\mathbb{a}_2) \geq \sup\{\mathbb{A}(\mathbb{x}_2) - \delta: \mathbb{x}_2 \in \mathbb{u}_2\}$. Then

$$\mathbb{A}(\mathbb{a}_1).\mathbb{A}(\mathbb{a}_2) \geq \sup\{\mathbb{A}(\mathbb{x}_1): \mathbb{x}_1 \in \mathbb{u}_1\}.\sup\{\mathbb{A}(\mathbb{x}_2): \mathbb{x}_2 \in \mathbb{u}_2\} - \varepsilon$$

Consequently,

$$\begin{aligned} g(\mathbb{A})(\mathbb{y}_1 * \mathbb{y}_2) &\geq \sup\{\mathbb{A}(\mathbb{x}_1).\mathbb{A}(\mathbb{x}_2): \mathbb{x}_1 \in \mathbb{u}_1, \mathbb{x}_2 \in \mathbb{u}_2\} \\ &\geq \sup\{\mathbb{A}(\mathbb{x}_1): \mathbb{x}_1 \in \mathbb{u}_1\}.\sup\{\mathbb{A}(\mathbb{x}_2): \mathbb{x}_2 \in \mathbb{u}_2\} \\ &= g(\mathbb{A})(\mathbb{y}_1).g(\mathbb{A})(\mathbb{y}_2) \end{aligned}$$

Then, $g(\mathbb{A})$ is a fuzzy dot completely closed BH-ideal of ξ.

Proposition 3-6:

Let $\mathbb{A}$ and $\mathbb{B}$ be two fuzzy dot completely closed BH-ideals of BH-algebra χ. The $\mathbb{A} \times \mathbb{B}$ is a fuzzy dot completely closed BH-ideal of $\chi \times \chi$.

Proof. Let $(\mathbb{x}_1, \mathbb{y}_1)$ and $(\mathbb{x}_2, \mathbb{y}_2) \in \chi \times \chi$. Then

$$\begin{aligned} (\mathbb{A} \times \mathbb{B})((\mathbb{x}_1, \mathbb{y}_1) * (\mathbb{x}_2, \mathbb{y}_2)) &= (\mathbb{A} \times \mathbb{B})(\mathbb{x}_1 * \mathbb{x}_2, \mathbb{y}_1 * \mathbb{y}_2) = \mathbb{A}(\mathbb{x}_1 * \mathbb{x}_2).\mathbb{B}(\mathbb{y}_1 * \mathbb{y}_2) \\ &\geq (\mathbb{A}(\mathbb{x}_1).\mathbb{A}(\mathbb{x}_2)).(\mathbb{B}(\mathbb{y}_1).\mathbb{B}(\mathbb{y}_2)) \\ &= (\mathbb{A}(\mathbb{x}_1).\mathbb{B}(\mathbb{y}_1)).(\mathbb{A}(\mathbb{x}_2).\mathbb{B}(\mathbb{y}_2)) \\ &= \mathbb{A} \times \mathbb{B}(\mathbb{x}_1, \mathbb{y}_1).\mathbb{A} \times \mathbb{B}(\mathbb{x}_2, \mathbb{y}_2) \end{aligned}$$

Hence, $\mathbb{A} \times \mathbb{B}$ is a fuzzy dot completely closed BH-ideal of $\chi \times \chi$.

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